FINITE DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS AT ROOTS OF UNITY

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Abstract. We describe explicitly the canonical map $\chi : \text{Spec } U_\varepsilon(\widetilde{\mathfrak{g}}) \to \text{Spec } Z_\varepsilon$, where $U_\varepsilon(\widetilde{\mathfrak{g}})$ is a quantum loop algebra at an odd root of unity ε. Here Z_ε is the center of $U_\varepsilon(\widetilde{\mathfrak{g}})$ and Spec R stands for the set of all finite–dimensional irreducible representations of an algebra R. We show that Spec Z_ε is a Poisson proalgebraic group which is essentially the group of points of G over the regular adeles concentrated at 0 and ∞. Our main result is that the image under χ of Spec $U_\varepsilon(\widetilde{\mathfrak{g}})$ is the subgroup of principal adeles.

Contents.

§0. Introduction.

The purpose of this paper is to study finite–dimensional irreducible representations of the quantum loop algebra $\widetilde{U}_\varepsilon = U_\varepsilon(\widetilde{\mathfrak{g}})$ at an odd root of unity ε. Here $\mathfrak{g}$ is a simple finite–dimensional Lie algebra over $\mathbb{C}$ and $\widetilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ is the associated loop algebra.

Denoting by Spec R the set of all finite–dimensional irreducible complex representations of an associative algebra R over $\mathbb{C}$ and by Z the center of R, we have (by Schur's lemma) the canonical map:

$$\chi : \text{Spec } R \to \text{Spec } Z. \tag{0.1}$$

(Recall that the value of χ on a representation $\sigma \in \text{Spec } R$ is defined by $\sigma(z) = \chi(\sigma)I$ for $z \in Z$.) If R is a finitely generated module over Z (which is the case for $R = U_\varepsilon(\mathfrak{g})$ [DC–K]) one knows that the map χ is surjective with finite fibers and, moreover, it is bijective over a Zariski open dense subset of Spec Z. In other words, at least "generically", Spec Z parametrizes the set of all irreducible finite–dimensional irreducible representations of R. This well–known observation was the starting point for a thorough (albeit incomplete) study of Spec $U_\varepsilon(\mathfrak{g})$ taken up in [DC–K], [DC–K–P1,2,3] and other papers.

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In the case when $R = \widetilde{U}_\varepsilon$ the situation is quite different since $\widetilde{U}_\varepsilon$ is not finitely generated over its center $Z = Z_\varepsilon$. The canonical map χ is not surjective and is not generically bijective. The main result of the present paper is the calculation of the image of χ in Spec Z_ε.

The first result (Proposition 2.3) provides a (infinite) set of generators of the algebra Z_ε. By general principles, Z_ε has a canonical structure of a Poisson algebra. Furthermore, we show that Z_ε is a Hopf subalgebra of the Hopf algebra $\widetilde{U}_\varepsilon$. (Recall that this isn't the case for $U_\varepsilon(\mathfrak{g})$.) Thus, Z_ε is a Poisson Hopf algebra, and using a "Frobenius homomorphism" we obtain that it is isomorphic to a certain Poisson Hopf algebra $\overline{U}_1$ independent of the odd root of unity ε (Corollary 3.2.1 and 3.2.2).

In the dual language, Spec Z_ε is a Poisson proalgebraic group. Our first key result (Theorem 5.3) is the construction of a Poisson group isomorphism

$$\pi : \text{Spec } Z_\varepsilon \to \Omega \tag{0.2}$$

with a Poisson proalgebraic group Ω described below. This result and its proof are similar to that in the "finite type" case given by [DC–K–P].

The group Ω is defined as follows. Let $\underline{G}$ be the connected simply connected algebraic group over $\mathbb{C}$ whose Lie algebra is $\mathfrak{g}$. Consider the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and let $\underline{N}_\pm$ and $\underline{H}$ be the closed algebraic subgroups of $\underline{G}$ with the Lie subalgebras $\mathfrak{n}_\pm$ and $\mathfrak{h}$ respectively. We denote by Ω the subgroup of the proalgebraic group $\tilde{\tilde{G}} = \underline{G}(\mathbb{C}[[t^{-1}]]) \times \underline{G}(\mathbb{C}[[t]])$ consisting of elements of the form $(hu_-(t^{-1}), h^{-1}u_+(t))$ where $h \in \underline{H}(\mathbb{C}),\ u_\pm(t^\pm) \in \underline{G}(\mathbb{C}[[t^\pm]])$ and $u_+(0) \in \underline{N}_+(\mathbb{C}), u_-(\infty) \in \underline{N}_-(\mathbb{C})$. The Poisson structure on Ω is defined by making use of a suitable Manin triple (as explained in §4.1). Here we note only that the symplectic leaves of this Poisson structure on Ω are connected components of the intersections of Ω with the orbits of the group $\underline{G}(\mathbb{C}[t,t^{-1}]) \times \underline{G}(\mathbb{C}[t,t^{-1}])$ acting on $\underline{G}(\mathbb{C}((t^{-1}))) \times \underline{G}(\mathbb{C}((t)))$ by $(k_1,k_2)\cdot(a,b) = (k_1ak_2^{-1}, k_1bk_2^{-1})$.

In order to describe the image $\mathcal{F}$ of the map π in Ω, introduce the following notation. Let $\mathbb{C}(t)_0$ be the subalgebra of the field of rational functions in the indeterminant t consisting of functions regular at 0 and at ∞. (This is a semilocal algebra.) We have an embedding $\mathbb{C}(t)_0 \hookrightarrow \mathbb{C}[[t^{-1}]] \times \mathbb{C}[[t]]$ by taking the power series expansions at ∞ and 0. Our second key result (Theorem 6.6) is that

$$\mathcal{F} = \Omega \cap \{(g,g) \in \tilde{\tilde{G}} \mid \text{Ad } g \in (\text{Ad } G)(\mathbb{C}(t)_0)\}. \tag{0.3}$$

In other words $\mathcal{F}$ is described as follows. Let $\mathcal{O}$ be the algebra of algebraic functions in t which are regular at 0 and ∞. Consider $g \in \underline{G}(\mathcal{O})$ such that Ad g is defined over $\mathbb{C}(t)_0 \subset \mathcal{O},\ g(0) \in \underline{HN}_+(\mathbb{C}),\ g(\infty) \in \underline{HN}_-(\mathbb{C})$ and the product of projections of $g(0)$ and $g(\infty)$ on $\underline{H}(\mathbb{C})$ equals 1. Consider the pair $(a,b) \in \underline{G}(\mathbb{C}[[t^{-1}]]) \times \underline{G}(\mathbb{C}[[t]])$ where a (resp. b) is the power series expansion of g at ∞ (resp. 0). Then $\mathcal{F}$ consists of all such pairs.

We prove (0.3) in two steps. First, we develop a theory of "diagonal" finite–dimensional irreducible representations of $\widetilde{U}_\varepsilon$. A representation σ is called *diagonal* if $\chi(\sigma) \in \Omega \cap (\underline{H}(\mathbb{C}[[t^{-1}]]) \times \underline{H}(\mathbb{C}[[t]]))$. We show that these representations are classifed by their "highest weights", which are, essentially, n–tuples ($n = \text{rank }\mathfrak{g}$) of rational functions $(R_1(t),\ldots,R_n(t))$ which are regular at 0 and at ∞ and such that $R_i(0)R_i(\infty) = 1$ for all i (Theorem 6.3).

Note that any finite–dimensional representation of $U_q(\widetilde{\mathfrak{g}})$ defined for generic q is diagonal when specialized to $q = \varepsilon$. Finite–dimensional irreducible representations of $U_q(\widetilde{\mathfrak{g}})$ were classified in [CP2] by rational functions of a very special form, in agreement with our results.

The second step consists of two parts. First we show that the elements of $\widetilde{U}_\varepsilon$ act "quasipolynomially" in a finite–dimensional representation, which implies the inclusion $\subset$ in (0.3). The reverse inclusion on the "diagonal" part follows from Theorem 6.3. To establish it on the "off diagonal" part we use the fact that along a symplectic leaf of Ω the representation theory of $\widetilde{U}_\varepsilon$ remains unchanged (cf. [DC–K–P2]).

We do not know a complete classification of finite–dimensional irreducible representations, even in the case of $U_\varepsilon(\widetilde{\mathfrak{sl}_2})$. It follows from our results that we get all central characters by considering irreducible subquotients of tensor products of evaluation representations. Finite–dimensional irreducible representations of $U_\varepsilon(\widetilde{\mathfrak{sl}_n})$ were studied in [T], but we do not understand its connection to our work.

Throughout the paper we denote by $R^\times$ the set of invertible elements of a ring R, with the exception that $\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$. We denote by $\mathbb{Z}_+$ the set of non–negative integers.

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§1. Notation.

1.1 Let $\mathfrak{g}$ be a simple finite–dimensional Lie algebra over $\mathbb{C}$. Choose a Cartan subalgebra $\mathfrak{h}$, let $\Delta \subset \mathfrak{h}^*$ be the set of roots and let $Q = \mathbb{Z}\Delta$ be the root lattice. Let

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$$

be the root space decomposition. For each root $\alpha \in \Delta$ there exists a unique coroot $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ such that $< \alpha, \alpha^\vee >= 2$. Let $\Delta^\vee \subset \mathfrak{h}$ be the set of coroots and let $Q^\vee = \mathbb{Z}\Delta^\vee$ be the coroot lattice. Let $P = \{\lambda \in \mathfrak{h}^* | \langle \lambda, Q^\vee \rangle \subset \mathbb{Z}\}$ be the weight lattice and let $P^\vee = \{\lambda \in \mathfrak{h} \mid \langle \lambda, Q \rangle \subset \mathbb{Z}\}$ be the coweight lattice.

Denote by $(.|.)$ the invariant bilinear symmetric form on $\mathfrak{g}$ (and the induced form on $\mathfrak{g}^*$) normalized by the condition that the square length of a short root equals 2. For $\alpha \in \Delta$ let $d_\alpha = \frac{1}{2}(\alpha|\alpha)$—this is a positive integer.

Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group, i.e. the group generated by reflections s_α ($\alpha \in \Delta$) defined by $s_\alpha(\lambda) = \lambda - < \lambda, \alpha^\vee > \alpha$.

Choose a subset of positive roots $\Delta_+ \subset \Delta$ and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots. Let $d_i = d_{\alpha_i}$ and let $a_{ij} =< \alpha_i^\vee, \alpha_j >= 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i)$ ($i, j = 1, \dots, n$) be the Cartan integers. Then the d_i are the relatively prime positive integers such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j = 1, \dots, n$. Let $\omega_1, \dots, \omega_n \in P$ (resp. $\omega_1^\vee, \dots, \omega_n^\vee \in P^\vee$) be the fundamental weights (resp. coweights) i.e. $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ (resp. $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{ij}$). Note that $\alpha_j = \sum_i a_{ij}\omega_i$ and $\alpha_j^\vee = \sum_i a_{ji}\omega_i^\vee$. We also let $s_i = s_{\alpha_i}$.

Let $\mathfrak{n}_\pm = \oplus_{\alpha\in\Delta_+}\mathfrak{g}_{\pm\alpha}$ be the opposite maximal nilpotent subalgebras of $\mathfrak{g}$. Choose Chevalley generators $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ ($i = 1, \dots, n$) such that $[e_i, f_i] = \delta_{ij}\alpha_i^\vee$.

Let $\underline{G}$ be the connected simply connected algebraic group over $\mathbb{C}$ with the Lie algebra $\mathfrak{g}$. Let $\underline{N}_\pm$ and $\underline{H}$ be the closed algebraic subgroups of $\underline{G}$ with the Lie subalgebras $\mathfrak{n}_\pm$ and $\mathfrak{h}$ respectively. As usual, we denote by $\underline{G}(R)$ the group of points of $\underline{G}$ over a commutative associative ring R. We let $G = \underline{G}(\mathbb{C})$, $N_\pm = \underline{N}_\pm(\mathbb{C})$, $H = \underline{H}(\mathbb{C})$.

Let $\mathcal{B}$ be the braid group on generators $T_1, \dots T_n$ associated to the Weyl group W. For $i = 1, \dots, n$ let:

$$t_i = (\exp f_i)(\exp -e_i)(\exp f_i) \in G. \tag{1.1.1}$$

One knows that the map $T_i \mapsto t_i$ extends to a homomorphism $\mathcal{B} \to G$ so that via the adjoint representation of G the action of $\mathcal{B}$ on $\mathfrak{g}$ satisfies:

$$T_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i(\alpha)}, \quad T_i|_\mathfrak{h} = s_i. \tag{1.1.2}$$

1.2 We proceed to define the associated "extended" (= "affine") objects. Let $\tilde{Q} = Q \oplus \mathbb{Z}\delta$ be a lattice of rank $n+1$ over $\mathbb{Z}$ with the symmetric bilinear form extending $(.|.)$ on Q by $(Q|\delta) = 0,\ (\delta|\delta) = 0$. Let θ be the highest root in $\Delta_+ \subset \Delta$ and let $d_0 = d_\theta$. Let $\alpha_0 = \delta - \theta$, so that $(\alpha_0|\alpha_0) = 2d_0$. Then the set of *affine simple roots* $\widetilde{\Pi} = \{\alpha_0\} \cup \Pi$ is a $\mathbb{Z}$–basis of the *affine root lattice* $\widetilde{Q}$. Note that the matrix $(a_{ij} = 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i))_{i,j=0}^n$ is the extended Cartan matrix of $\mathfrak{g}$. Note also that $d_i a_{ij} = d_j a_{ji}$ for all $i, j = 0, \ldots, n$.

The *affine root system* is the set $\widetilde{\Delta} = \widetilde{\Delta}^{\mathrm{re}} \cup \widetilde{\Delta}^{\mathrm{im}}$, where

$$\widetilde{\Delta}^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}, \quad \widetilde{\Delta}^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}^\times\}.$$

We let $\widetilde{\Delta}_+ = \widetilde{\Delta}^{\mathrm{re}}_+ \cup \widetilde{\Delta}^{\mathrm{im}}_+$, where

$$\widetilde{\Delta}^{\mathrm{re}}_+ = \{\alpha + n\delta \mid \alpha \in \Delta,\ n \in \mathbb{N}\} \cup \Delta_+, \quad \widetilde{\Delta}^{\mathrm{im}}_+ = \{n\delta \mid n \in \mathbb{N}\}.$$

Note that $\widetilde{\Delta}_+ = \widetilde{Q}_+ \cap \widetilde{\Delta}$, where $\widetilde{Q}_+ = \sum_{j=0}^n \mathbb{Z}_+ \alpha_j$.

The action of W on Q is extended to $\widetilde{Q}$ by letting $W(\delta) = \delta$. Define the reflection s_0 of $\widetilde{Q}$ by $s_0(\alpha) = s_\theta(\alpha) + \langle \alpha, \theta^\vee \rangle\, \delta$. The *affine Weyl group* $\widetilde{W}$ is then the subgroup of $GL(\widetilde{Q})$ generated by all $s_i,\ i = 0, \ldots, n$. Recall that $\widetilde{W}$ is a Coxeter group on generators $\{s_0, \ldots, s_n\}$. Let $\mathcal{T}$ denote the group of all permutations σ of the index set $\{0, 1, \ldots, n\}$ such that $a_{\sigma(i),\sigma(j)} = a_{ij}\ (i, j = 1, \ldots, n)$. This group acts by automorphisms of the lattice $\widetilde{Q}$ by $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ which preserve the bilinear form $(.|.)$. Consider the extended affine Weyl group $\widetilde{W}^e = \mathcal{T} \ltimes \widetilde{W}$. The group $P^\vee$ imbeds in $\widetilde{W}^e$ via $\alpha \mapsto t_\alpha$, where

$$t_\alpha(\beta) = \beta - (\beta|\alpha)\delta\ (\beta \in \widetilde{Q}).$$

Recall that $Q^\vee$ then imbeds in $\widetilde{W}$ so that $\widetilde{W} = W \ltimes Q^\vee$.

Let $\widetilde{\mathcal{B}}$ denote the braid group on generators $T_0, \ldots, T_n$ associated to $\widetilde{W}$, and form the extended braid group $\widetilde{\mathcal{B}}^e = \mathcal{T} \ltimes \widetilde{\mathcal{B}}$ in the obvious way. For $\sigma w \in \widetilde{W}^e$ and a reduced expression $w = s_{i_1} \ldots s_{i_k}$ we let $T_{\sigma w} = \sigma T_{i_1} \ldots T_{i_k}$. This is independent of the choice of the reduced expression.

1.3 The "extended" objects are related to the loop algebra and the loop group in the following well–known way (cf. [K]). Let $\mathbb{C}((t))$ denote the field of Laurent series in t, and let $\mathbb{C}[[t]]$ and $\mathbb{C}[t, t^{-1}]$ be its subrings of formal power series and of Laurent polynomials. In what follows, $\mathfrak{g}((t)),\ \mathfrak{g}[[t]]$, and $\mathfrak{g}[t, t^{-1}]$ stand for $\mathbb{C}((t)) \otimes_{\mathbb{C}} \mathfrak{g},\ \mathbb{C}[[t]] \otimes_{\mathbb{C}} \mathfrak{g}$, and $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$. Similarly, we denote by $\underline{G}((t)),\ \underline{G}[[t]]$, and $\underline{G}[t, t^{-1}]$ respectively the groups of points of the algebraic group G over $\mathbb{C}((t)), \mathbb{C}[[t]]$, and $\mathbb{C}[t, t^{-1}]$.

We let $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}],\ \widetilde{G} = \underline{G}[t, t^{-1}]$ be the *loop algebra* and the *loop group*. We note that $\mathfrak{g} \cong 1 \otimes \mathfrak{g}$ is a subalgebra of $\tilde{\mathfrak{g}}$ and G is a subgroup of $\widetilde{G}$.

The *root space decomposition* of $\tilde{\mathfrak{g}}$ is defined as follows:

$$\tilde{\mathfrak{g}} = \mathfrak{h} \oplus (\oplus_{\alpha \in \tilde{\Delta}} \mathfrak{g}_\alpha),$$

where $\mathfrak{g}_{\alpha + k\delta} = t^k \otimes \mathfrak{g}_\alpha\ (\alpha \in \Delta, k \in \mathbb{Z})$, $\mathfrak{g}_{k\delta} = t^k \otimes \mathfrak{h}\ (k \in \mathbb{Z}^\times)$.

Choose $e_\theta \in \mathfrak{g}_\theta$ and $e_{-\theta} \in \mathfrak{g}_{-\theta}$ such that $[e_\theta, e_{-\theta}] = -\theta^\vee$, and let $e_0 = t \otimes e_{-\theta} \in \mathfrak{g}_{\alpha_0}, f_0 = t^{-1} \otimes e_\theta \in \mathfrak{g}_{-\alpha_0}$. Then $e_i,\ f_i\ (i = 0, \ldots, n)$ are the *Chevalley generators* of $\tilde{\mathfrak{g}}$. Along with $\mathfrak{h}$ they satisfy the well–known collection of defining relations [K].

Let $t_0 = (\exp f_0)(\exp -e_0)(\exp f_0) \in \widetilde{G}$. Then (as in the finite–dimensional case) the map $T_i \mapsto t_i$ extends to a homomorphism $\widetilde{\mathcal{B}} \to \widetilde{G}$ so that (1.1.2) holds for all $i = 0, \dots, n$ [KP].

1.4 Recall the following construction of the set $\widetilde{\Delta}_+^{\mathrm{re}}$ [Be2, Pa]. Fix an element $x \in Q^\vee \subset \widetilde{W}$ such that $\langle x, \alpha_i \rangle > 0$ for all $i = 1, \dots n$, and fix a reduced expression $x = s_{j_1} \dots s_{j_d}$ (in the Coxeter group $\widetilde{W}$).

Let $(i_k)_{k\in\mathbb{Z}}$ be the sequence of integers such that $i_k = j_{k \bmod (d)}$. Then the following two important properties hold:

(1) The roots $\beta_k := \begin{cases} s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_{i_k}), & k \geq 0, \\ s_{i_0}s_{i_{-1}}\dots s_{i_{k+1}}(\alpha_{i_k}), & k < 0 \end{cases}$ comprise $\widetilde{\Delta}_+^{\mathrm{re}}$.

(2) Each subsection $s_{i_k}s_{i_{k+1}}\dots s_{i_{l-1}}s_{i_l}$ for $k < l$ is reduced.

This definition allows a total order $<$ to be defined on the set of positive roots $\widetilde{\Delta}_+$ given by:

$$\beta_0 < \beta_{-1} < \beta_{-2} < \dots < r\delta < s\delta < \dots < \beta_2 < \beta_1 \quad \text{if } r < s. \tag{1.4.1}$$

Remark 1.4. We give the following example for $\widetilde{U}_q(\widetilde{\mathfrak{sl}_3})$. Pick $x = 2\rho \in Q^\vee$ where $2\rho = \sum_{\alpha\in\Delta_+} \alpha$. Then a reduced expression of 2ρ is given by $(s_0s_1s_2s_1)^2$ and the ordering (1.4.1) has the form:

$$\begin{gathered}\delta-\alpha_1-\alpha_2 < \delta-\alpha_2 < 2\delta-\alpha_1-\alpha_2 < \delta-\alpha_1 < 3\delta-\alpha_1-\alpha_2 < 2\delta-\alpha_2 < 4\delta-\alpha_1-\alpha_2 \\ < 2\delta-\alpha_1 < \dots < k\delta < \dots < \delta+\alpha_1+\alpha_2 < \alpha_2 < \alpha_1+\alpha_2 < \alpha_1\end{gathered}$$

This order is convex in the sense that if $\alpha \in \widetilde{\Delta}_+^{\mathrm{re}}$ and $\beta \in \widetilde{\Delta}_+$ are such that $\alpha + \beta \in \widetilde{\Delta}_+$, then $\alpha < \alpha+\beta < \beta$. ([Pa]). The following elements form a basis of the vector space spanned by the real root spaces of $\tilde{\mathfrak{g}}$:

$$\begin{aligned} e_{\beta_k} &= \begin{cases} t_{i_0}\dots t_{i_{k+1}}(e_{i_k}), & k \leq 0 \\ t_{i_1}t_{i_2}\dots t_{i_{k-1}}(e_{i_k}), & k > 0 \end{cases} \\ e_{-\beta_k} &= \begin{cases} t_{i_0}\dots t_{i_{k+1}}(f_{i_k}), & k \leq 0 \\ t_{i_1}t_{i_2}\dots t_{i_{k-1}}(f_{i_k}), & k > 0 \end{cases} \end{aligned} \tag{1.4.2}$$

We remark that when $\beta_k = \alpha + k\delta$ for $\alpha \in \widetilde{\Delta}^{\mathrm{re}}$ we have $e_{\alpha+k\delta} = t^k \otimes e_\alpha$.

1.5 One defines the *quantum loop algebra* $\widetilde{U}_q = U_q(\tilde{\mathfrak{g}})$ of Drinfel′d and Jimbo as an algebra over $\mathbb{C}(q)$ on generators $E_i,\ F_i\ (i = 0, \dots, n),$ and $K_\alpha\ (\alpha \in P)$, subject to the following relations:

$$\begin{aligned} &[K_\alpha, K_\beta] = 0,\ K_\alpha K_\beta = K_{\alpha+\beta},\ K_0 = 1, \\ &K_\alpha E_j K_\alpha^{-1} = q^{(\alpha|\alpha_j)}E_j,\ K_\alpha F_j K_\alpha^{-1} = q^{-(\alpha|\alpha_j)}F_j, \\ &[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \\ &(\mathrm{ad}_q E_i)^{1-a_{ij}}(E_j) = 0, \quad (\mathrm{ad}_q F_i)^{1-a_{ij}}(F_j) = 0 \ \text{ if } i \neq j. \end{aligned} \tag{1.5.1}$$

We make frequent use of the abbreviated notation $K_i = K_{\alpha_i}$, $[s]_d = \frac{q^{ds}-q^{-ds}}{q^d-q^{-d}}$, $[s]_d! = [1]_d \dots [s]_d$, $\left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right]_d = \frac{[m]_d!}{[r]_d![m-r]_d!}$, and we write $[s]$ for $[s]_1$. The notation ad_q is explained below (see (1.5.2)).

We recall that $\widetilde{U}_q$ has a Hopf algebra structure with comultiplication Δ, antipode S and counit η defined by:

$$
\begin{aligned}
&\Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i,\ \Delta F_i = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i,\ \Delta K_\alpha = K_\alpha \otimes K_\alpha,\\
&SE_i = -K_{-\alpha_i}E_i,\ SF_i = -F_iK_i,\ SK_\alpha = K_{-\alpha},\\
&\eta E_i = 0,\ \eta F_i = 0,\ \eta K_\alpha = 1.
\end{aligned}
$$

Then in (1.5.1) and in what follows we define ad_q by:

$$(\mathrm{ad}_q x)(y) = \sum_i a_i y S(b_i) \text{ if } \Delta x = \sum_i a_i \otimes b_i. \tag{1.5.2}$$

Introduce the $\mathbb{C}$–algebra anti-automorphism κ of U_q, defined by:

$$\kappa(E_i) = F_i, \quad \kappa(F_i) = E_i, \quad \kappa(K_\alpha) = K_{-\alpha}, \quad \kappa(q) = q^{-1}.$$

Recall that the braid group $\widetilde{\mathcal{B}}$ acts as a group of automorphisms of the algebra $\widetilde{U}_q$ by the following formulas [L]:

$$
\begin{aligned}
&T_iE_i = -F_iK_i,\ T_i(E_j) = \frac{(-1)^{a_{ij}}}{[a_{ij}]_{d_i}!}(ad_qE_i)^{-a_{ij}}(E_j) \text{ if } \ i \neq j,\\
&T_iK_\alpha = K_{s_i\alpha} \quad (\alpha \in P), \quad \kappa T_i = T_i\kappa.
\end{aligned}
$$

This action is extended to the action of $\widetilde{\mathcal{B}}^e$ in the obvious way.

Let $\widetilde{U}_q^+$ (resp. $\widetilde{U}_q^-$) denote the subalgebra of $\widetilde{U}_q$ generated by the E_i (resp. F_i), and let U_q^0 be the subalgebra generated by the K_α ($\alpha \in P$). Then multiplication defines an isomorphism ([Ro, L]):

$$\widetilde{U}_q^- \otimes U_q^0 \otimes \widetilde{U}_q^+ \xrightarrow{\sim} \widetilde{U}_q. \tag{1.5.3}$$

Define the subalgebras $\widetilde{U}_q^{\geq 0}$ (resp. $\widetilde{U}_q^{\leq 0}$) generated by U_q^0 and the E_i (resp. F_i). The algebras $\widetilde{U}_q^+$ and $\widetilde{U}_q^{\geq 0}$ are graded by $\tilde{Q}_+$ in the usual way:

$$\widetilde{U}_q^{+\ (\text{resp.}\geq 0)} = \oplus_\nu (\widetilde{U}_q^{+\ (\text{resp.}\geq 0)})_\nu.$$

1.6 For each $\beta_k \in \widetilde{\Delta}_+^{\mathrm{re}}$ define the *root vector* E_{β_k} by:

$$E_{\beta_k} = \begin{cases} T_{i_0}^{-1} \dots T_{i_{k+1}}^{-1}(E_{i_k}) & \text{if } k \leq 0 \\ T_{i_1}T_{i_2} \dots T_{i_{k-1}}T_{i_k}(F_{i_k}) & \text{if } k > 0 \end{cases} \tag{1.6.1}$$

Remark 1.6. A useful property of these real root vectors is that each (up to a factor from U_q^0) is some integral power of T_x ($x = s_{i_0}s_{i_1} \dots s_{i_d} \in Q^\vee$) applied to the finite set $S = \{T_{i_d}^{-1} \dots T_{i_{k+1}}^{-1}E_{i_k} \mid 0 \leq k \leq d\}$.

Definition 1.6. *For $i = 1, \ldots, n$ and $m > 0$ let*

$$\psi_m^{(i)} = K_i^{-1}[E_i, E_{m\delta-\alpha_i}], \tag{1.6.2}$$

$$\psi_{-m}^{(i)} = \kappa(\psi_m^{(i)}),\ \ \psi_0^{(i)} = \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}. \tag{1.6.3}$$

For $k > 0$, define *imaginary root vectors* $E_{k\delta}^{(i)},\ (i = 1, \ldots, n)$ by the following functional equation involving the $\psi_k^{(i)}$:

$$\exp\big((q^{d_i} - q^{-d_i})\sum_{k=1}^{\infty} E_{k\delta}^{(i)} u^k\big) = 1 + (q^{d_i} - q^{-d_i})\sum_{k=1}^{\infty} \psi_k^{(i)} u^k. \tag{1.6.4}$$

As usual we extend these definitions to $\widetilde{U}_q^{\leq 0}$ using the antiautomorphism $\kappa :\quad E_{-\beta} := \kappa(E_\beta)$ for $\beta \in \widetilde{\Delta}_+$.

These root vectors have the nice property that up to a sign they coincide with the Drinfel′d generators [D]. Namely, we have:

Theorem 1.6 [Be]**.** *The algebra $\widetilde{U}_q$ is an associative algebra on generators ($i = 1, \ldots, n$, $k \in \mathbb{Z}$):*

$$E_\beta\ (\beta = \pm\alpha_i + k\delta \in \widetilde{\Delta}),\ \ E_{k\delta}^{(i)}\ (k \neq 0),\ \ K_\alpha\ (\alpha \in P),$$

and the following relations:

(1.6.5a) $[K_\alpha, E_{k\delta}^{(i)}] = [K_\alpha, K_\beta] = 0,\ \ K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha|\beta)} E_\beta,$

(1.6.5b) $[E_{k\delta}^{(i)}, E_{l\delta}^{(j)}] = 0,\ \ [E_{k\delta}^{(i)}, E_{\pm\alpha_j + l\delta}] = \pm\varepsilon_{ij}^{kl} \frac{1}{k}[k a_{ij}]_{d_i} E_{\pm\alpha_j+(k+l)\delta},$

(1.6.5c) $E_{\pm\alpha_i+(k+1)\delta} E_{\pm\alpha_j+l\delta} - q^{\pm(\alpha_i|\alpha_j)} E_{\pm\alpha_j+l\delta} E_{\pm\alpha_i+(k+1)\delta}$
$$= \varepsilon_{ij}^{kl}(q^{\pm(\alpha_i|\alpha_j)} E_{\pm\alpha_j+(l+1)\delta} E_{\pm\alpha_i+k\delta} - E_{\pm\alpha_i+k\delta} E_{\pm\alpha_j+(l+1)\delta}),$$

(1.6.5d) $[E_{\alpha_i+k\delta}, E_{-\alpha_j+l\delta}] = \delta_{ij} K_i^{\mathrm{sgn}(k+l)} \psi_{k+l}^{(i)},$

(1.6.5e) $Sym_{k_1,k_2,\ldots,k_m} \sum_{r=0}^{m} (-1)^r \varepsilon_{ij}^{\vec{k}l} \left[{m \atop r} \right]_{d_i}$
$$E_{\pm\alpha_i+k_1\delta} \ldots E_{\pm\alpha_i+k_r\delta} E_{\pm\alpha_j+l\delta} E_{\pm\alpha_i+k_{r+1}\delta} \ldots E_{\pm\alpha_i+k_m\delta} = 0,$$

where $i \neq j$, $m = 1 - a_{ij}$, and $\varepsilon_{ij}^{kl} = \pm 1$ as explained in [Be]. *Sym denotes symmetrization with respect to the indices $\vec{k} = (k_1,\ k_2, \ldots k_m)$. The function sgn$(x)$ is defined to be $\frac{|x|}{x}$ for $x \neq 0$ and sgn$(0) = 0$.* $\square$

1.7 As defined, the root vectors E_β ($\beta \in \widetilde{\Delta}_+$) are in $\widetilde{U}_q^{\geq 0}$. Defining $\dot{E}_{\alpha+k\delta} = E_{\alpha+k\delta}$ (resp. $= K_{-\alpha} E_{-\alpha+k\delta}$) if $\alpha \in \Delta_+$ and $\dot{E}_{k\delta} = E_{k\delta}$, we have $\dot{E}_\beta \in \widetilde{U}_q^+$ for all $\beta \in \widetilde{\Delta}_+$. Introduce the monomials $M_{(a_\beta)} = \Pi_{\tilde{\Delta}_+,<} \dot{E}_\beta^{a_\beta} \in \widetilde{U}_q^+$. Here $(a_\beta) \in {\mathbb{Z}_+}^{\tilde{\Delta}_+}$, where ${\mathbb{Z}_+}^{\tilde{\Delta}_+}$ denotes the set of maps $f : \tilde{\Delta}_+ \mapsto \mathbb{Z}_+$ with finite support and $<$ denotes that the product is convexly ordered. Let $N_{(a_\beta)} := \kappa(M_{(a_\beta)}) \in \widetilde{U}_q^-$.

Proposition 1.7. [Be2] (a) *The $M_{(a_\beta)}$ form a basis of $\widetilde{U}_q^+$ over $\mathbb{C}(q)$.*

(b) *The elements* $N_{(a_\beta)}K_\alpha M_{(a'_\beta)}$*, where* $(a_\beta),(a'_\beta)\in {\mathbb{Z}_+}^{\tilde\Delta_+}$*,* $\alpha\in P$*, form a basis of* $\widetilde{U}_q$ *over* $\mathbb{C}(q)$*.*

(c) *Let* $\alpha,\beta\in\widetilde{\Delta}_+$ *be such that* $\beta>\alpha$*. Then:*

$$\dot E_\beta \dot E_\alpha - q^{(\alpha|\beta)}\dot E_\alpha \dot E_\beta = \sum_{\alpha<\gamma_1<\dots<\gamma_n<\beta} c_{\vec\gamma}\dot E_{\gamma_1}^{a_1}\dots \dot E_{\gamma_n}^{a_n}.$$

where $c_{\vec\gamma}\in\mathbb{C}[q,q^{-1}]$ *for* $\vec\gamma=(\gamma_1,\gamma_2,\dots,\gamma_n),\in\widetilde{\Delta}_+^n$*.* $\square$

1.8 Using the above PBW type basis we define a filtration on $\widetilde{U}_q$ as in [DC–K]. Consider a monomial $N_{(a_\beta)}K_\alpha M_{(a'_\beta)}$ where $a_\beta,a'_\beta\in{\mathbb{Z}_+}^{\tilde\Delta_+}$ and $\alpha\in P$. Define its total height by

$$d_0(N_{(a_\beta)}K_\alpha M_{(a'_\beta)})=\sum_\beta (a_\beta+a'_\beta)\mathrm{ht}\ \beta,$$

and its total degree by

$$d(N_{(a_\beta)}K_\alpha M_{(a'_\beta)})=(d_0(N_{(a_\beta)}K_\alpha M_{(a'_\beta)}),(a_\beta),(a'_\beta))\in\mathbb{Z}^{2\tilde\Delta_++1}.$$

We view ${\mathbb{Z}_+}^{2\tilde\Delta_++1}$ as a totally ordered semigroup with the usual lexicographical order.

Introduce a ${\mathbb{Z}_+}^{2\tilde\Delta_++1}$–filtration of the algebra $\widetilde{U}_q$ by letting U_s ($s\in{\mathbb{Z}_+}^{2\tilde\Delta_++1}$) be the span of the monomials $N_{(a_\beta)}K_\alpha M_{(a'_\beta)}$ such that $d(N_{(a_\beta)}K_\alpha M_{(a'_\beta)})\le s$. Proposition 1.7 implies:

Proposition 1.8. *The associated graded algebra Gr* $\widetilde{U}_q$ *of the* ${\mathbb{Z}_+}^{2\tilde\Delta_++1}$*–filtered algebra* $\widetilde{U}_q$ *is an algebra over* $\mathbb{C}(q)$ *on generators* E_α ($\alpha\in\widetilde{\Delta}_+$ *counting multiplicities*) *and* K_β ($\beta\in P$) *subject to the following relations:*

(1.8.1)

$$\begin{aligned}
&K_\alpha K_\beta=K_{\alpha+\beta},\ K_0=1;\\
&K_\alpha E_\beta=q^{(\alpha|\beta)}E_\beta K_\alpha,\ K_\alpha F_\beta=q^{-(\alpha|\beta)}F_\beta K_\alpha;\\
&E_\alpha E_{-\beta}=E_{-\beta}E_\alpha,\ \textit{if}\ \alpha,\beta\in\widetilde{\Delta}_+;\\
&E_\alpha E_\beta=q^{(\alpha|\beta)}E_\beta E_\alpha,\ E_{-\alpha}E_{-\beta}=q^{(\alpha|\beta)}E_{-\beta}E_{-\alpha},\ \textit{if}\ \alpha,\beta\in\widetilde{\Delta}_+\ \textit{and}\ \beta<\alpha.\ \square
\end{aligned}$$

1.9 Fix a primitive ℓ–th root of unity ε. Let $\mathcal{A}^\varepsilon$ be the subring of $\mathbb{C}(q)$ consisting of rational functions regular at $q=\varepsilon$. Let $\widetilde{U}_{\mathcal{A}^\varepsilon}$ be the $\mathcal{A}^\varepsilon$–subalgebra of $\widetilde{U}_q$ generated by the elements $E_i,F_i,K_i^{\pm1}$ and $\psi_0^{(i)}$. Let $(q-\varepsilon)\widetilde{U}_{\mathcal{A}^\varepsilon}$ be the 2–sided ideal generated by $(q-\varepsilon)$ in $\widetilde{U}_{\mathcal{A}^\varepsilon}$. Define the algebra $\widetilde{U}_\varepsilon$ over $\mathbb{C}$, the *specialization* of $\widetilde{U}_q$ at ε, by $\widetilde{U}_\varepsilon=\widetilde{U}_{\mathcal{A}^\varepsilon}/(q-\varepsilon)\widetilde{U}_{\mathcal{A}^\varepsilon}$.

Remark 1.9. The algebra $\widetilde{U}_\varepsilon$ is the associative algebra over $\mathbb{C}$ on generators $E_i,\ F_i,\ (i=0,\dots,n), K_\alpha$ ($\alpha\in P$) and defining relations (1.5.1) where q is replaced by ε, provided that $\ell>max_i(d_i)$. Of course, all relations (1.6.5) with q replaced by ε hold in $\widetilde{U}_\varepsilon$. Also, Gr $\widetilde{U}_\varepsilon$ is the algebra over $\mathbb{C}$ obtained by substituting q for ε in (1.8.1).

1.10 We make explicit the above formulas for $U_q(\widetilde{\mathfrak{sl}_2})$ (cf. [Da]). In this case $\Delta_+=\{\alpha\}$, $\omega=\frac12\alpha$ is the only fundamental weight, and there is a unique choice of the sequence $(i_k)_{k\in\mathbb{Z}}$,

namely $i_k = k(\mathrm{mod}\ 2)$ (see [Be2]). Then $\beta_k = \alpha - k\delta$ for $k \le 0$ and $\beta_k = -\alpha + k\delta$ for $k > 0$. Define the generators $E_{\alpha-k\delta},\ E_{-\alpha+k\delta},\ E_{k\delta}$ as in formulas (1.6.2–5). Then $\widetilde{U}_q = U_q(\widetilde{\mathfrak{sl}_2})$ is the algebra over $\mathbb{C}(q)$ on generators $K_\omega^{\pm 1}, E_{\pm\alpha+k\delta}$ $(k \in \mathbb{Z})$, and $E_{k\delta}$ $(k \in \mathbb{Z}^\times)$ with defining relations:

(1.10.1)
$$
\begin{aligned}
&(a)\ [K_\omega, E_{k\delta}] = 0,\ K_\omega E_\beta K_\omega^{-1} = q^{(\omega|\beta)} E_\beta\ (\beta = \pm\alpha + k\delta),\\
&(b)\ [E_{k\delta}, E_{\pm\alpha+l\delta}] = \pm\frac{1}{k}[2k] E_{\pm\alpha+(k+l)\delta},\ [E_{k\delta}, E_{l\delta}] = 0,\\
&(c)\ E_{\pm\alpha+(k+1)\delta} E_{\pm\alpha+l\delta} - q^{\pm 2} E_{\pm\alpha+l\delta} E_{\pm\alpha+(k+1)\delta}\\
&\qquad\qquad = q^{\pm 2} E_{\pm\alpha+(l+1)\delta} E_{\pm\alpha+k\delta} - E_{\pm\alpha+k\delta} E_{\pm\alpha+(l+1)\delta},\\
&(d)\ [E_{\alpha+k\delta}, E_{-\alpha+l\delta}] = K_\omega^{2\mathrm{sgn}(k+l)} \psi_{k+l},
\end{aligned}
$$

where $\psi_m = \psi_m^{(1)}$ are defined by (1.6.2–3).

§2. The center Z_ε of $\widetilde{U}_\varepsilon$.

2.1 It is clear from (1.6.5b) that before proceeding to the calculation of Z_ε we need to calculate the quantity $\Delta_k :=\det([k a_{ij}]_{d_i})_{i,j=1}^n$.

Lemma 2.1. *The determinants Δ_k are given by the following formulas:*

$$
\begin{array}{llll}
A_n: & [k]^{n-1}[(n+1)k] & B_n: & [k]_2^{n-1}[k][2]_{(2n-1)k}\\
C_n: & [k]^{n-1}[k]_2[2]_{(n+1)k} & D_n: & [k]^{n-1}[2k][2]_{(n-1)k}\\
E_6: & [k]^5[3k]([2]_{4k}-1) & E_7: & [k]^6[2k]([2]_{6k}-1)\\
E_8: & [k]^8([2]_{8k}+[2]_{6k}-[2]_{2k}-1) & F_4: & [k]^2[k]_2^2([2]_{6k}-1)\\
G_2: & [k][3k]_3([2]_{10k}+[2]_{8k}-[2]_{2k}-1).\ \square & &
\end{array}
$$

Corollary 2.1. *Let $q = \varepsilon$ be a primitive ℓ–th root of 1 where ℓ is odd. Then $\Delta_k \ne 0$ for all non–zero integers k such that $\ell \nmid k$ provided that the following conditions on ℓ hold:*

(2.1.1)
$$
\begin{aligned}
&A_n \text{ and } C_n: \mathit{gcd}(\ell, n+1) = 1;\\
&B_n: \mathit{gcd}(\ell, 2n-1) = 1;\ D_n: \mathit{gcd}(\ell, n-1) = 1;\\
&E_6 \text{ and } G_2: \mathit{gcd}(\ell, 3) = 1.\ \square
\end{aligned}
$$

Remark 2.1. The conditions (2.1.1) on ℓ follows in all cases except for G_2 from the condition $\gcd(\ell, h^\vee) = 1$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ [K, Chapter 6].

2.2 From now on we shall assume that ℓ is an odd integer greater than 1 satisfying the conditions (2.1.1). Fix a primitive ℓ–th root of unity ε. We turn now to the calculation of the center Z_ε of $\widetilde{U}_\varepsilon$.

Lemma 2.2 (a) *Let $\beta \in \widetilde{\Delta}^{\mathrm{re}}$. Then $E_\beta^\ell \in Z_\varepsilon$ (for any $\ell > \max_i d_i$).*

(b) *Let $i = 1, \dots, n,\ k \in \mathbb{Z}^\times$. Then $E_{k\ell\delta}^{(i)} \in Z_\varepsilon$ (for any $\ell > 1$).*

Proof. (a) As shown in [DC–K–P2], $(\mathrm{ad}_\varepsilon E_i)^\ell(E_j) = E_i^\ell E_j - E_j E_i^\ell = 0$ by the last of relations (1.5.1). Similarly one checks that the F_i^ℓ are central. Now the other real root vectors are braid group translates of the E_i or F_i. (b) follows from (1.6.5a,b). $\square$

Since the algebra $\mathrm{Gr}\,\widetilde{U}_\varepsilon$ is quasipolynomial over $\mathbb{C}$, its center $\overline{Z}_\varepsilon$ can be calculated using the methods of [DC–K–P2]. Let $A = (a_{\beta,\beta'})$, where $a_{\beta,\beta'} = -a_{\beta',\beta} = (\beta|\beta')$ if $\beta < \beta' \in \widetilde{\Delta}_+$ and $a_{\beta,\beta} = 0$ (so that A is antisymmetric). Let $B = ((\omega_i|\beta))$ where $i = 1, \ldots, n$ and $\beta \in \widetilde{\Delta}_+$. Form the infinite matrix indexed by the set $M = \widetilde{\Delta} \cup \{1, \ldots, n\}$:

$$S = \begin{pmatrix} A & -{}^tB & 0 \\ B & 0 & -B \\ 0 & {}^tB & -A \end{pmatrix}$$

Its matrix elements are the commutation coefficients of the algebra $\mathrm{Gr}\,\widetilde{U}_\varepsilon$ in the ordered basis given by Proposition 1.7. Consider the range of S mod ℓ, i.e. $S : \mathbb{Z}^M \to (\mathbb{Z}/\ell)^M$ and let H_S be the kernel of this map. Then as in [DC–K–P2, Proposition 3.3], a basis for $\overline{Z}_\varepsilon$ is given by $\{\prod_< E_\alpha^{h_\alpha} \mid h = (h_\alpha) \in H_S \cap \mathbb{Z}_+^M\}$. Given a basis of H_S we obtain a polynomial basis of $\mathrm{Gr}\, Z_\varepsilon$. Such a basis of H_S can be made apparent (see [DC–K–P2]) by finding the elementary divisors of the matrix

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ -A & 0 & {}^tB \\ 0 & 0 & 0 \end{pmatrix}$$

over $\mathbb{Z}[\frac{1}{2}]$.

We bring S_1 to a matrix S' which will have the same elementary divisors using the following row operations on A to obtain a matrix A':

(1) $\mathrm{Row}(-\alpha + k\delta) - \mathrm{Row}(-\alpha + (k+1)\delta) \to \mathrm{Row}(-\alpha + k\delta)$, and
(2) $\mathrm{Row}(\alpha + (k-1)\delta) - \mathrm{Row}(\alpha + k\delta) \to \mathrm{Row}(\alpha + (k-1)\delta)$.

Then

$$A' = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_2 \end{pmatrix} \text{ and } S' = \begin{pmatrix} 0 & 0 & 0 \\ -A' & 0 & {}^tB' \\ 0 & 0 & 0 \end{pmatrix}$$

where T_1 is upper triangular and T_2 is lower triangular with the diagonal elements of T_1 and T_2 equal 1. Hence the kernel is generated by ℓ–th powers of the real root vectors and the imaginary root vectors. Thus, we have proved

Proposition 2.2. *$\overline{Z}_\varepsilon$ is generated by K_α^ℓ $(\alpha \in P)$, E_β^ℓ $(\beta \in \widetilde{\Delta}^{\mathrm{re}})$ and $E_{k\delta}^{(i)}$ $(i = 1, \ldots, n, k \in \mathbb{Z}^\times)$.* $\square$

2.3 We apply the previous considerations to calculating the center Z_ε of U_ε. Z_ε inherits a filtration from U_ε, and it is straightforward that $\mathrm{Gr}\, Z_\varepsilon \subset \overline{Z}_\varepsilon$.

Lemma 2.3.1. *Let ℓ satisfy the conditions* (2.1.1). *Let $P(E_{k\delta}^{(i)})$ be a polynomial in the $E_{k\delta}^{(i)}$ $(k > 0)$ where $\Delta_k \neq 0$ for some k. Then there exists j, $1 \leq j \leq n$, for which $[P(E_{k\delta}^{(i)}), E_{-\alpha_j+\delta}] \neq 0$.*

Proof. For notational convenience denote by $c_k^{i,j}$ the q–coefficient $[ka_{ij}]$ evaluated at ε. Given a monomial $\prod_{m=1}^D E_{k_m\delta}^{(i_m)}$ we have the following formula

$$(2.3.1)\qquad [\prod_{m=1}^{D} E_{k_m\delta}^{(i_m)}, E_{-\alpha_j+\delta}] = \sum_{\substack{S\subset\{1,\dots,D\}\\ S\neq\emptyset}} \prod_{m\in S} c_{k_m}^{i,j} \prod_{m'\in S^c} E_{k_{m'}\delta}^{(i_{m'})} E_{-\alpha_j+(\sum_{m\in S} k_m+1)\delta}.$$

which is calculated from (1.6.5).

Write each monomial $\prod_{m=1}^D E_{k_m\delta}^{(i_m)}$ in P in non–decreasing order with respect to the $k_m \geq 1$. Without loss of generality assume that each monomial in $P(E_{k\delta}^{(i)})$ has a factor $E_{k\delta}^{(i)}$ such that $\ell \nmid k$. Pick such a monomial for which D is maximal and k_1 is minimal. Summing over all monomials with the same $k_1, k_2, \dots, k_D$ and the same $i_2, \dots, i_D$ we see that $P(E_{k\delta}^{(i)})$ is of the form

$$(2.3.2)\qquad (\sum_{r=1}^{n} a_r E_{k_1\delta}^{(i_r)}) E_{k_2\delta}^{(i_2)} \dots E_{k_D\delta}^{(i_D)}$$
$$+ \text{ other algebraically independent expressions.}$$

By the assumption of the lemma there exists a $E_{-\alpha_{j_1}+\delta}$ for which $[\sum_r a_r E_{k_1\delta}^{(r)}, E_{-\alpha_{j_1}+\delta}] = \sum_r c_{k_1}^{r,j_1} (E_{-\alpha_{j_1}+(k_1+1)\delta}) \neq 0$ and so

$$[P(E_{k\delta}^{(i)}), E_{-\alpha_{j_1}+\delta}] = \sum_r a_r c_{k_1}^{r,j} \prod_{m=2}^{D} E_{k_m}^{(i_m)} E_{-\alpha_{j_1}+k_1\delta}$$
$$+ \text{other algebraically independent expressions} \neq 0. \quad \square$$

Lemma 2.3.2. *Suppose that ℓ is an odd integer greater than 1 satisfying* (2.1.1)*. Then Gr Z_ε is the subalgebra of $\overline{Z}_\varepsilon$ generated by $K_\alpha^\ell, E_\beta^\ell, E_{j\ell\delta}^{(i)}$ for $\alpha \in P,\ \beta \in \widetilde{\Delta}^{\mathrm{re}}, j \in \mathbb{Z}^\times, i = 1, \dots, n$.*

Proof. Certainly the above generators are in Gr Z_ε. Take an expression $N_{(a_\beta)} K_\alpha^\ell M_{(a'_\beta)} \in$ Gr Z_ε which is not homogeneous with respect to the inherited grading. By assumption we can find $x \in \widetilde{U}_\varepsilon$ so that $d(x) < d(N_{(a_\beta)} K_\alpha^\ell M_{(a'_\beta)})$ and

$$(2.3.3)\qquad N_{(a_\beta)} K_\alpha^\ell M_{(a'_\beta)} + x \in Z_\varepsilon.$$

We show that neither $F_{k\delta}^{(i)}$ or $E_{k\delta}^{(i)}$ appears in $N_{(a_\beta)} K_\alpha^\ell M_{(a'_\beta)}$ where ℓ does not divide k. We can assume without loss of generality that such an $E_{k\delta}^{(i)}$ appears in $M_{(a_\beta)}$. Collect from x all monomials that have the same factors in the PBW basis as $N_{(a_\beta)} K_\alpha^\ell$ and the real root vectors of $M_{(a'_\beta)}$. Then (2.3.3) can be written as

$$(2.3.4)\qquad N_{(a_\beta)} K_\alpha^\ell M_{(a'_\beta),\beta\leq 0} P(E_{k\delta}^{(i)}) M_{(a'_\beta),\beta>0} + x'.$$

where P is a polynomial in $E_{k\delta}^{(i)}$ for $i=1,\dots,n,\ k>0$. We also know that

$$[N_{(a_\beta)}K_\alpha^\ell M_{(a'_\beta),\beta\le 0}P(E_{k\delta}^{(i)})M_{(a'_\beta),\beta>0}+x',E_{-\alpha_j}]=0. \tag{2.3.5}$$

where $d(x')<d(N_{(a_\beta)}K_\alpha^\ell M_{(a'_\beta),\beta\le 0}P(E_{k\delta}^{(i)})M_{(a'_\beta),\beta>0})$, $j=1\dots n$. Since $[M_{(a'_\beta),\beta\le 0},E_{-\alpha_j}]=[M_{(a'_\beta),\beta>0},E_{-\alpha_j}]=0$, using the grading on N^- it follows that $[P(E_{k\delta}^{(i)}),E_{-\alpha_j}]=0$ for each $j=1,\dots,n$. By Lemma 2.3.1 it follows that $\ell|k_m$ for each $E_{k_m\delta}^{(i_m)}$. $\square$

We have shown the following:

Proposition 2.3. *Let ε be a primitive ℓ–th root of unity, where ℓ is as in Lemma 2.3.2. Then the center Z_ε of $\widetilde{U}_\varepsilon$ is generated by the elements E_β^ℓ ($\beta\in\widetilde{\Delta}^{\rm re}$), $E_{j\ell\delta}^{(i)}$ ($j\in\mathbb{Z}^\times, i=1,\dots,n$), K_α^ℓ ($\alpha\in P$).* $\square$

Remark 2.3. Proposition 2.3 also holds for the central extension $U_\varepsilon(\hat{\mathfrak{g}})$ of $U_\varepsilon(\tilde{\mathfrak{g}})$ (where we add the central element C of $U_\varepsilon(\hat{\mathfrak{g}})$).

§3. The Frobenius isomorphism and the Poisson structure on Z_ε.

3.1 It is known that Z_ε is a Poisson algebra as explained in [DC–K–P], a Poisson structure being given by the formula:

$$\{x,y\}=\frac{[\tilde{x},\tilde{y}]}{\ell(q^\ell-q^{-\ell})} \bmod (q-\varepsilon). \tag{3.1.1}$$

Here $\tilde{x}$ stands for a preimage of $x\in Z_\varepsilon\subset\widetilde{U}_\varepsilon$ under the specialization map.

Lemma 3.1. *Let $i=1,\dots,n$, then $\{E_i^\ell,E_{\delta-\alpha_i}^\ell\}=K_i^\ell\ell(\varepsilon-\varepsilon^{-1})E_{k\ell\delta}^{(i)}$.*

Proof. We consider the case of $\widetilde{\mathfrak{s}l_2}$, the general case follows from this case and (1.6.5c). From §2 we know that $E_\alpha^\ell,\ E_{\delta-\alpha}^\ell$ are central. From Proposition 2.3 we see

$$[E_\alpha^\ell,E_{\delta-\alpha}^\ell]=(q-\varepsilon)\sum c_{(k,k',k'')}\prod_{<,k\in K}E_{\alpha+k\delta}^\ell P(E_{k'\ell\delta})\prod_{<,k''\in K''}E_{-\alpha+k''\delta}^\ell+(q-\varepsilon)^2y. \tag{3.1.2}$$

Using relation 1.7c) we see that if $k\neq 0$ for for some $k\in K$ then $k''\neq 0$ for some $k''\in K''$. This is impossible since $k+k''$ must equal 1. Therefore the bracket (3.1.2) must be purely imaginary and central and therefore, by Prop. 2.3, a multiple of $E_{\ell\delta}$. The coefficient is easily calculated to be $\ell(\varepsilon-\varepsilon^{-1})$ by using (1.10.1d) and (1.6.4). $\square$

3.2 In this section we introduce a renormalized version of $\widetilde{U}_q$ which on specialization to 1 is isomorphic as a Poisson algebra to Z_ε. In the finite type case this variation was considered in the works of [Re] and [DC–P].

Definition 3.2 (a). Let $\overline{U}_{\mathcal{A}^1}$ be the subalgebra of $\widetilde{U}_q$ generated by:

$$\overline{E}_\alpha=(q^{d_i}-q^{-d_i})E_\alpha,\quad K_\beta\ (\alpha\in\widetilde{\Delta},\ \beta\in P).$$

(b) Let $\overline{U}_1=\overline{U}_{\mathcal{A}^1}/(q-1)\overline{U}_{\mathcal{A}^1}$.

$\overline{U}_1$ has a Poisson structure given by (cf. (3.1.1)):

$$\{x,y\}=\frac{[\tilde{x},\tilde{y}]}{(q-q^{-1})} \bmod (q-1).$$

Lemma 3.2.1. *$\overline{U}_1$ is a commutative algebra. As a Poisson algebra $\overline{U}_1$ is generated by $\overline{E}_i, \overline{F}_i, K_\alpha, i = 0, \dots, n,\ \alpha \in P$.*

Proof. See [DC–K–P], [DC–P]. $\square$

Lemma 3.2.2. *There exists a unique Poisson algebra homomorphism $Fr : \overline{U}_1 \to Z_\varepsilon$ such that $\overline{E}_\alpha \mapsto E_\alpha^\ell$, for $\alpha \in \widetilde{\Delta}^{\text{re}}$.*

Proof. $\{\overline{E}_i |\ i = 0, 1, \dots, n\}$ form a set of Poisson algebra generators for $\overline{U}_1^+$ with the defining relations $\text{ad}_P^{1-a_{ij}} \overline{E}_i(\overline{E}_j) = 0$, where ad_P denotes the Poisson adjoint action. Let Fr be defined as in the lemma. Using [L, Theorem 35.1.8] it follows that Fr is well defined as a Poisson algebra homomorphism. Since as defined this map is compatible with the braid group action [L, 41.1.9], Fr satisfies the conditions of the lemma. Since $\overline{U}_1$ is generated as a Poisson algebra by $E_\alpha, \alpha \in \widetilde{\Delta}^{\text{re}}$, Fr is necessarily unique. $\square$

Corollary 3.2.1. *Fr is a Hopf algebra homomorphism.*

Proof. It is an easy calculation that $\Delta(E_i^\ell) = K_i^\ell \otimes E_i^\ell + E_i^\ell \otimes 1,\ \Delta(F_i^\ell) = F_i^\ell \otimes K_i^{-\ell} + 1 \otimes F_i^\ell$, and $\Delta(K_i^\ell) = K_i^\ell \otimes K_i^\ell$. Thus Fr and Δ commute on $\overline{E}_i, \overline{F}_i, K_i$, which are Poisson generators of $\overline{U}_1$. Since coproduct and Poisson bracket commute the statement follows. $\square$

Furthermore we have:

Lemma 3.2.3. *Let $k > 0$. Then $Fr(\overline{E}_{k\delta}) = \ell(\varepsilon - \varepsilon^{-1}) E_{k\ell\delta}$.*

Proof. We have an explicit set of polynomial generators for both algebras and the lemma will follow by calculating the images of the imaginary root vectors under Fr. It follows from [CP] that modulo the ideal in $\overline{U}_1$ generated by the real root vectors, $\overline{E}_{k\delta}$ is primitive for each $k \in \mathbb{Z}$. For the same reason (and since the center is closed under coproduct by the previous Corollary) it follows that $E_{k\ell\delta}$ is primitive modulo the ℓ-th powers of the real root vectors. Since Fr and coproduct commute, it follows that $Fr(\overline{E}_{k\delta})$ is primitive (modulo the real root vectors). Now it follows (as in Lemma 3.1) that $Fr(K\overline{E}_{k\delta}) = Fr(\{E_\alpha, E_{k\delta-\alpha}\}) = K^\ell P(E_{k'\ell\delta})_{k' \le k}$. Since $P(E_{k'\ell\delta})_{k' \le k}$ is primitive, it must be a multiple of $E_{k\ell\delta}$. It follows using 1.10.1(d) and 1.6.4 that this coefficient is $\ell(\varepsilon - \varepsilon^{-1})$. $\square$

Corollary 3.2.2. *$Fr : \overline{U}_1 \to Z_\varepsilon$ is an isomorphism of Hopf Poisson algebras.*

In general, Lemma 3.2.3 implies

Lemma 3.2.4. *$Fr(1 + \sum_{k=1}^\infty \overline{\psi}_k^{(i)} t^k) = \exp(\ell(\varepsilon^{d_i} - \varepsilon^{-d_i}) \sum_{k=1}^\infty E_{k\ell}^{(i)} t^k)$ for $i = 1 \dots n$.* $\square$

3.3 Introduce the notation:

$$X_i^- = -\sum_{k=-\infty}^{+\infty} \overline{E}_{-\alpha_i + k\delta} t^k, \quad X_i^+ = \sum_{k=-\infty}^{+\infty} \overline{E}_{\alpha_i + k\delta} t^k,$$

$$X_{>a,i}^\pm = \pm \sum_{k>a} \overline{E}_{\pm\alpha_i + k\delta} t^k, \quad X_{\le a,i}^\pm = \pm \sum_{k \le a} \overline{E}_{\pm\alpha_i + k\delta} t^k, \text{ etc.},$$

$$\Psi_i = 1 + \sum_{k=1}^\infty \overline{\psi}_k^{(i)} t^k, \quad \Phi_i = 1 - \sum_{k=1}^\infty \overline{\psi}_{-k}^{(i)} t^{-k}, \quad i = 1, \dots, n.$$

We record the following calculations using the Poisson structure on $\overline{U}_1$ (see [Be] 4.1, 4.7 for the relevant commutation formulas) :

$$\begin{aligned}
&\{\overline{E}_j,\Phi_i\} = (\alpha_j|\alpha_i)\Phi_i X^+_{<0,j}, && \{\overline{E}_j,\Psi_i\} = -(\alpha_j|\alpha_i)\Psi_i X^+_{>0,j},\\
&\{\overline{F}_j,\Phi_i\} = -(\alpha_j|\alpha_i)\Phi_i X^-_{<0,j}, && \{\overline{F}_j,\Psi_i\} = (\alpha_j|\alpha_i)\Psi_i X^-_{>0,j},\\
&\{\overline{E}_i,K_i\} = -d_i\overline{E}_iK_i, && \{\overline{F}_i,K_i\} = d_i\overline{F}_iK_i,\\
&\{\overline{E}_i,X^+_{\geq 0,i}\} = -d_iX^+_{\geq 0,i}X^+_{>0,i}, && \{\overline{E}_i,X^+_{\leq 0,i}\} = d_iX^+_{\leq 0,i}X^+_{<0,i},\\
&\{\overline{E}_i,X^-_{>0,i}\} = d_i(K_i(\Psi-1)), && \{\overline{E}_i,X^-_{\leq 0,i}\} = d_i(K_i-K_i^{-1}\Phi_i),\\
&\{\overline{F}_i,X^-_{\leq 0,i}\} = -d_iX^-_{\leq 0,i}X^-_{<0,i}, && \{\overline{F}_i,X^-_{>0,i}\} = d_iX^-_{\geq 0,i}X^-_{>0,i},\\
&\{\overline{F}_i,X^+_{\geq 0,i}\} = d_i(K_i^{-1}-K_i\Psi_i), && \{\overline{F}_i,X^+_{<0,i}\} = d_i(K_i^{-1}\Phi_i-K_i^{-1}),\\
&\{\overline{F}_i,X^+_i\} = d_i(K_i^{-1}\Phi_i-K_i\Psi_i).
\end{aligned} \tag{3.3.1}$$

Since Φ and Ψ are series that start with 1, fractional powers of these series are well defined. Let $(\overline{a}_{ij}) = ({}^tA)^{-1}$, then $\omega_i = \sum_j \overline{a}_{ij}\alpha_j$. Define $\Phi_{\omega_i} = \prod_j \Phi_j^{\overline{a}_{ij}}$ (resp. $\Psi_{\omega_i} = \prod_j \Psi_j^{\overline{a}_{ij}}$). The following identities hold:

$$\begin{aligned}
&\{\overline{E}_j,\Phi_{\omega_i}\} = (\omega_i|\alpha_j)\Phi_{\omega_i}X^+_{<0,j}, && \{\overline{E}_j,\Psi_{\omega_i}\} = -(\omega_i|\alpha_j)\Psi_{\omega_i}X^+_{>0,j},\\
&\{\overline{F}_j,\Phi_{\omega_i}\} = -(\omega_i|\alpha_j)\Phi_{\omega_i}X^-_{<0,j}, && \{\overline{F}_j,\Psi_{\omega_i}\} = (\omega_i|\alpha_j)\Psi_{\omega_i}X^-_{>0,j}.
\end{aligned} \tag{3.3.2}$$

§4. The Poisson proalgebraic group Ω.

4.1 Consider the group (cf. 1.3):

$$\tilde{\tilde{G}} = \underline{G}((t^{-1})) \times \underline{G}((t))$$

and introduce subgroups Ω and K of $\tilde{\tilde{G}}$ as follows.

Let $\tilde{N}_- = \{g(t^{-1}) \in \underline{G}[[t^{-1}]] \mid g(\infty) \in N_-\}$, $\tilde{N}_+ = \{g(t) \in \underline{G}[[t]] \mid g(0) \in N_+\}$. Let $\Omega = \{(hu_-, h^{-1}u_+) \mid u_\pm \in \tilde{N}_\pm, h \in H\}$, $K = \{(g,g) \mid g \in \underline{G}[t,t^{-1}]\}$.

The Lie algebra of $\tilde{\tilde{G}}$ is $\tilde{\tilde{\mathfrak{g}}} := \mathfrak{g}((t^{-1})) \oplus \mathfrak{g}((t))$. The Lie subalgebra Lie $\Omega \subset \tilde{\tilde{\mathfrak{g}}}$ of Ω consists of pairs $(a_1(t^{-1}), a_2(t)) \in \tilde{\tilde{\mathfrak{g}}}$, where $a_1(t^{-1}) \in \mathfrak{g}[[t^{-1}]], a_2(t) \in \mathfrak{g}[[t]]$ are such that $a_1(\infty) = n_- + h, a_2(0) = n_+ - h$ and $n_\pm \in \mathfrak{n}_\pm, h \in \mathfrak{h}$. The Lie algebra Lie K consists of pairs (a,a), where $a \in \mathfrak{g}[t,t^{-1}]$. We have

$$\tilde{\tilde{\mathfrak{g}}} = \text{Lie } \Omega \oplus \text{Lie } K, \tag{4.1.1}$$

where $\oplus$ is the direct sum of vector spaces. The invariant bilinear symmetric form $(.|.)$ on the Lie algebra $\mathfrak{g}$ extend bilinearly to a $\mathbb{C}((t^{-1}))$(resp. $\mathbb{C}((t))$)–valued form on $\mathfrak{g}((t^{-1}))$ (resp. $\mathfrak{g}((t))$). We denote by $(.|.)_\infty$ (resp. $(.|.)_0$) the constant term. This is a $\mathbb{C}$–valued invariant bilinear symmetric form on $\mathfrak{g}((t^{-1}))$ (resp. $\mathfrak{g}((t))$). Define a ($\mathbb{C}$–valued) invariant bilinear symmetric form $(.|.)$ on $\tilde{\tilde{\mathfrak{g}}}$ by:

$$((x_1,x_2)|(y_1,y_2)) = -(x_1|y_1)_\infty + (x_2|y_2)_0. \tag{4.1.2}$$

then the subalgebras Lie Ω and Lie K of $\tilde{\tilde{\mathfrak{g}}}$ are isotropic with respect to the form (4.1.2). Thus $(\tilde{\tilde{\mathfrak{g}}}, \text{Lie } \Omega, \text{Lie } K)$ is a Manin triple. This endows the proalgebraic group Ω with a canonical structure of a Poisson proalgebraic group (see e.g. [DC–K–P3, §4]).

The general description of symplectic leaves (given e.g. by [DC–K–P3, Proposition 4.2]) implies the following:

Proposition 4.1. *Consider the following action of the group $K \times K$ on $\tilde{\tilde{G}}$:*

$$((a,a),(b,b)) \cdot (g_1, g_2) = (ag_1b^{-1}, ag_2b^{-1}). \tag{4.1.3}$$

Then the symplectic leaves of Ω are connected components of intersections of orbits of this action with $\Omega \subset \tilde{\tilde{G}}$. $\square$

Note that the restriction of the canonical map

$$\alpha : \Omega \to \tilde{\tilde{G}}/K$$

is a finite covering of some open set of $\tilde{\tilde{G}}/K$. Considering the left action of K on $\tilde{\tilde{G}}/K$, an element $e \in \text{Lie } K$ defines a vector field on $\tilde{\tilde{G}}/K$ which using α can be lifted to Ω. We shall denote again by e the resulting vector field on Ω. All these vector fields are tangent to the symplectic leaves of Ω.

4.2 For each $i = 0, \ldots, n$ there exists a unique normal subgroup $\tilde{N}_+^{(i)}$ (resp. $\tilde{N}_-^{(i)}$) of $\tilde{N}_+$ (resp. $\tilde{N}_-$) such that $\tilde{N}_+ = \tilde{N}_+^{(i)} \ltimes \exp(\mathbb{C}e_i)$, $\tilde{N}_- = \tilde{N}_-^{(i)} \ltimes \exp(\mathbb{C}f_i)$. This allows us to define regular functions $\tilde{x}_i$ and $\tilde{y}_i$ on $\tilde{N}_+$ and $\tilde{N}_-$ respectively by letting:

$$u_+ = u_+^{(i)} \exp(-\tilde{x}_i e_i), \ u_- = u_-^{(i)} \exp(\tilde{y}_i f_i), \text{ where } u_\pm^{(i)} \in \tilde{N}_\pm^{(i)}.$$

Since $\mathbb{C}[H] = P$, any $\alpha \in P$ defines a regular function on H, which we denote by $\tilde{z}_\alpha$. We extend these functions to regular functions on Ω by letting $\tilde{x}_i$, $\tilde{y}_i$ and $\tilde{z}_\alpha$ be defined at the point $(h^{-1}u_-, hu_+)$ by $\tilde{x}_i(u_+)$, $\tilde{y}_i(u_-)$, and $\alpha(h)$ respectively.

As in [DC–K–P] the braid group acts on Ω by the formula

$$\tilde{T}_i(hu_-, h^{-1}u_+) = (t_i h u_-^{(i)} \exp(\tilde{x}_i e_i) t_i^{-1}, t_i h (\exp \tilde{y}_i f_i) h^{-2} u_+^{(i)} t_i^{-1}).$$

For an element a of a Poisson algebra A we denote by P_a the derivation of A defined by $P_a(x) = \{a, x\}$. In the same way as in [DC–K–P2, § 7.6] the following theorem is proved.

Theorem 4.2 (a) *The functions $\tilde{x}_i$, $\tilde{y}_i$, $(i = 0, \ldots n)$ and $\tilde{z}_\alpha$ $(\alpha \in P)$ generate the coordinate ring $\mathbb{C}[\Omega]$ as a Poisson algebra.*

(b) $\Delta\tilde{x}_i = 1 \otimes \tilde{x}_i + \tilde{x}_i \otimes \tilde{z}_{-\alpha_i}$, $\Delta\tilde{y}_i = 1 \otimes \tilde{y}_i + \tilde{y}_i \otimes \tilde{z}_{-\alpha_i}$, $\Delta\tilde{z}_\alpha = \tilde{z}_\alpha \otimes \tilde{z}_\alpha$.

(c) $P_{\tilde{z}_{\alpha_i}\tilde{x}_i} = -d_i \tilde{z}_{\alpha_i}(f_i, f_i)$, $P_{\tilde{z}_{\alpha_i}\tilde{y}_i} = d_i \tilde{z}_{\alpha_i}(e_i, e_i)$, $P_{\tilde{z}_{\alpha_i}} = \frac{1}{2} d_i \tilde{z}_{\alpha_i}(\alpha_i^\vee, \alpha_i^\vee)$.

(d) *The $\tilde{T}_i$ define a map $\tilde{\mathcal{B}} \to Aut(\Omega)$ (where Aut denotes Poisson algebraic variety automorphisms).* $\square$

§5. The isomorphism π : Spec $Z_\varepsilon \to \Omega$.

5.1 Consider the following (closed proalgebraic) subgroups of the group $\tilde{N}_+$:

$$\tilde{N}_+^+ = \prod_{k \le 0} \exp(\mathbb{C}e_{\beta_k}), \ \tilde{N}_+^- = \prod_{k>0} \exp(\mathbb{C}e_{\beta_k}), \ \tilde{N}_+^0 = \prod_{i=1}^n \prod_{k \ge 1} \exp(\mathbb{C}\omega_k^{\vee(i)}),$$

where we let $\omega_k^{\vee(i)} = t^k \otimes \omega_i^\vee$, and similarly those of $\tilde{N}_-$:

$$\tilde{N}_-^+ = \prod_{k>0} \exp(\mathbb{C}e_{-\beta_k}), \ \tilde{N}_-^- = \prod_{k \le 0} \exp(\mathbb{C}e_{-\beta_k}), \tilde{N}_+^0 = \prod_{i=1}^n \prod_{k \ge 1} \exp(\mathbb{C}\omega_{-k}^{\vee(i)}).$$

Then multiplication establishes isomorphisms of proalgebraic varieties:

$$\tilde{N}_{\pm} \simeq \tilde{N}^{-}_{\pm} \times \tilde{N}^{0}_{\pm} \times \tilde{N}^{+}_{\pm}. \tag{5.1.1}$$

Define functions $\tilde{x}_{\beta_k}$ and $\tilde{x}^{(i)}_k$ by letting

$$\tilde{x}_{\beta_k}(\prod_j \exp \gamma_j e_{\beta_j}) = \gamma_k,\ \tilde{x}^{(i)}_k(\prod_j \exp \gamma^{(i)}_j \omega^{\vee(i)}_j) = \gamma^{(i)}_k.$$

Then the coordinate ring of the group $\tilde{N}^{+(\text{resp. }-)}_{+}$ is the polynomial algebra $\mathbb{C}[\tilde{x}_{\beta_k} \mid k > 0 \text{ (resp. } k \le 0)]$, and similar statements holds for the groups $\tilde{N}^{\pm}_{-}$. Finally, the coordinate ring of $\tilde{N}^{0}_{\pm}$ is the algebra $\mathbb{C}[\tilde{x}^{(i)}_{\pm k}, (k > 0)]$.

5.2 For $\alpha \in \Delta$ and $i = 1, \dots, n$ introduce constants:

$$d_\alpha = \frac{1}{2}(\alpha|\alpha),\ c_\alpha = (\varepsilon^{d_\alpha} - \varepsilon^{-d_\alpha})^{\ell},\ b_i = \ell(\varepsilon^{d_i} - \varepsilon^{-d_i}).$$

Consider the following elements of Z_ε:

$$z_\beta = K^{\ell}_{\beta}\ (\beta \in P);\ x_{-\alpha+k\delta} = c_\alpha E^{\ell}_{-\alpha+k\delta}\ (\alpha \in \Delta_+,\ k \in \mathbb{Z}),$$
$$x_{\alpha+k\delta} = -c_\alpha z_{-\alpha} E^{\ell}_{\alpha+k\delta}\ (\alpha \in \Delta_+,\ k \in \mathbb{Z}); x^{(i)}_{k\delta} = b_i E^{(i)}_{\ell k\delta}\ (k \in \mathbb{Z}^{\times}).$$

Introduce the subalgebras $Z^0_0, Z^+_+,\ Z^0_+$ and Z^-_+ of Z_ε generated by the elements z_β ($\beta \in P$); x_{β_k} ($k > 0$); $x_{-\beta_k}$ ($k \ge 0$); and $x^{(i)}_{k\delta}$ ($i = 1, \dots, n,\ k \in \mathbb{Z}^{\times}$). Similarly introduce the subalgebras Z^+_-, Z^0_-, and Z^-_- of Z_ε. By Theorem 5.2, Z_ε is isomorphic to the tensor product of these subalgebras. Hence defining the subalgebras $Z_+ = Z^-_+ \otimes Z^0_+ \otimes Z^+_+$, and $Z_- = Z^-_- \otimes Z^0_- \otimes Z^+_-$ we have:

$$Z_\varepsilon = Z_- \otimes Z^0_0 \otimes Z_+. \tag{5.2.1}$$

As usual, given a commutative associative algebra A over $\mathbb{C}$ we denote by Spec A the proalgebraic variety of all algebra homomorphisms $A \to \mathbb{C}$. Note that given a proalgebraic variety X, defining a regular map $\rho : \text{Spec } A \to X$ amounts to giving an element of $X(A)$, which we denote by the same letter ρ.

We let

$$\begin{aligned}
\pi^+_+ &= \prod_{k \le 0, <} \exp(-x_{\beta_k} e_{\beta_k}) \in \tilde{N}^+_+(Z^+_+),\ \pi^+_- = \prod_{k > 0, <} \exp(z^2_{\beta_k} x_{-\beta_k} e_{-\beta_k}) \in \tilde{N}^+_-(Z^+_-),\\
\pi^-_+ &= \prod_{k > 0, <} \exp(-x_{\beta_k} e_{\beta_k}) \in \tilde{N}^-_+(Z^-_+),\ \pi^-_- = \prod_{k \le 0, <} \exp(x_{-\beta_k} e_{-\beta_k}) \in \tilde{N}^-_-(Z^-_-),\\
\pi^0_+ &= \prod_{i=1}^{n} \prod_{k > 0} \exp(x^{(i)}_{k\delta} t^k \omega^\vee_i) \in \tilde{N}^0_+(Z^0_+),\ \pi^0_- = \prod_{i=1}^{n} \prod_{k > 0} \exp(-x^{(i)}_{-k\delta} t^{-k} \omega^\vee_i) \in \tilde{N}^0_-(Z^0_-).
\end{aligned} \tag{5.2.2}$$

As previously remarked, (5.2.2) defines maps $\pi^+_+ : \text{Spec } Z^+_+ \to \tilde{N}^+_+$, etc. Finally we define the map $\pi^0_0 : \text{Spec } Z^0_0 \to H$ by identifying the function α on H with the element $z_\alpha \in Z^0_0$. We may write π^0_0 in the form

$$\pi^0_0 = \sum_i z_{\omega_i} \otimes \alpha^\vee_i = \sum_i z_{\alpha_i} \otimes \omega^\vee_i \in Z^0_0 \otimes_{\mathbb{Z}} Q^\vee = \underline{H}(Z^0_0). \tag{5.2.3}$$

We point out that π_+^0 (resp. π_-^0) is the image under the Frobenius correspondence of the map $\sum_i \Psi_{\omega_i} \otimes \alpha_i^\vee$ (resp. $\sum_i \Phi_{\omega_i} \otimes \alpha_i^\vee$). This is since

$$\sum_i \mathrm{Fr}(\Psi_{\omega_i}) \otimes \alpha_i^\vee = \prod_{i=1}^{n} \prod_{k\geq 0} \prod_{j=1}^{n} \exp(x_{k\delta}^{(j)} t^k \alpha_i^\vee)^{\overline{a}_{ij}} = \prod_{i,j} \prod_k \exp(\overline{a}_{ij} x_{k\delta}^{(j)} \alpha_i^\vee)$$
$$= \prod_{i,j} \prod_k \exp(x_{k\delta}^{(j)} \overline{a}_{ij} \alpha_i^\vee) = \prod_j \prod_k \exp(x_{k\delta}^{(j)} \omega_j^\vee).$$

We define the maps:

$$\pi_+ = \pi_+^+ \times \pi_+^0 \times \pi_+^- : \text{Spec } Z_+ \to \tilde{N}_+, \tag{5.2.4}$$
$$\pi_- = \pi_-^+ \times \pi_-^0 \times \pi_-^- : \text{Spec } Z_- \to \tilde{N}_-.$$

Using (5.2.1), we write an element of Spec Z_ε in the form $u_- h u_+$, where $u_\pm \in$ Spec $Z_\pm, h \in$ Spec Z_0^0. Now we may define the isomorphism of proalgebraic varieties

$$\pi : \text{Spec } Z_\varepsilon \xrightarrow{\sim} \Omega$$

by letting

$$\pi(u_- h u_+) = \big((\pi_0^0(h))^{-1} \pi_-(u_-), \pi_0^0(h) \pi_+(u_+)\big). \tag{5.2.5}$$

Remark 5.2. We have written π in the form (5.2.5) so that its relationship to the finite type map [DC–K–P] is apparent. In later consideration of finite dimensional representations it will be useful to express π in a form where $(\pi_0^0)^{\pm 1}$ is incorporated into π_- and π_+ in the first and second factors respectively.

5.3 The following is our first key result.

Theorem 5.3. *The map π is an isomorphism of Poisson proalgebraic groups which commutes with the action of $\tilde{\mathcal{B}}$.*

Proof of this theorem is along the same lines as the analogous result for the finite type case in [DC–K–P]. It is based on the same simple lemma:

Lemma 5.3. *[DC–K–P2, Lemma 7.2] Let A and B be two commutative Poisson Hopf algebras and let $\varphi : A \to B$ be an algebra isomorphism compatible with the augmentation maps. Suppose that elements $a_1, \dots, a_s$ generate A as a Poisson algebra and that the following two properties hold:*

(i) $(\varphi \otimes \varphi)\Delta_A(a_i) = \Delta_B \varphi(a_i),\ i = 1, \dots, s;$

(ii) $\{\varphi(a_i), \varphi(a)\} = \varphi(\{a_i, a\}),\ i = 1, \dots, s, a \in A.$

Then φ is an isomorphism of Poisson Hopf algebras. $\square$

We apply this lemma to Poisson Hopf algebras $A = \mathbb{C}[\Omega],\ B = Z_\varepsilon$, and the map $\varphi = \pi^*$. For the Poisson generators of A we take elements $\tilde{x}_i, \tilde{y}_i,\ (i = 0, \dots, n)$ and $\tilde{z}_\alpha$ $(\alpha \in P)$ (cf. Theorem 4.2(a)). Note that $\varphi(\tilde{x}_i) = x_i,\ \varphi(\tilde{y}_i) = y_i$, and $\varphi(\tilde{z}_\alpha) = z_\alpha$ $(\alpha \in P)$, that φ is compatible with augmentation maps and that the assumption (i) of Lemma 5.3 obviously

holds (cf. Theorem 4.2(b)). Hence, in view of Theorem 4.2(c), in order to check assumption (ii) of Lemma 5.3 we have to show that for $i = 0, \dots, n$:

$$P_{z_{\alpha_i} x_i} = -d_i z_{\alpha_i}(f_i, f_i),\ P_{z_{\alpha_i} y_i} = d_i z_{\alpha_i}(e_i, e_i),\ P_{z_{\alpha_i}} = \frac{1}{2} d_i z_{\alpha_i}(\alpha_i^\vee, \alpha_i^\vee), \tag{5.3.1}$$

where the vector fields (f_i, f_i), etc. on Spec Z_ε are the pull–backs of the corresponding vector fields on Ω via the map φ.

In order to prove (5.3.1) consider the map

$$\gamma((a, b)) = a^{-1} b,\ (a, b) \in \Omega.$$

Then the action (4.1.3) of K on $\tilde{\tilde{G}}$ induces the action by conjugation of $\underline{G}[t, t^{-1}]$ on $\gamma(\Omega)$. Since the fibers of γ are finite, it suffices to check the pushdowns of the equalities (5.3.1) to $\gamma(\Omega)$. It is easy to see that the latter equalities are as follows ($i = 0, \dots, n$):

$$P_{z_{\alpha_i} x_i} = -d_i z_{\alpha_i} f_i,\ P_{z_{\alpha_i} y_i} = d_i z_{\alpha_i} e_i,\ P_{z_{\alpha_i}} = \frac{1}{2} d_i z_{\alpha_i} \alpha_i^\vee. \tag{5.3.2}$$

Consider a faithful finite–dimensional representation of the group G. Then the meaning of, for example, the first equality of (5.3.2) is interpreted as follows. The left–hand side is the Poisson bracket of $z_{\alpha_i} x_i$ with all elements of the matrix

$$M := (\pi_-^- \pi_-^0 \pi_-^+)^{-1} (\pi_0^0)^2 (\pi_+^- \pi_+^0 \pi_+^+). \tag{5.3.3}$$

The right–hand side is the usual bracket $[-d_i z_{\alpha_i} f_i, M]$ (since $\underline{G}[t, t^{-1}]$ acts by conjugation).

We now explain how to perform these calculations assuming the affine rank 2 case, which will be calculated in the next section making use of the Frobenius map. Recall that (5.3.2) holds for the finite type case (this is the main result of [DC–K–P]).

Let $Z_\varepsilon^{0,\mathrm{im}}$ be the subalgebra of Z_ε generated by z_β and $x_{k\delta}^{(i)}$ ($\beta \in Q^\vee$, $k \in \mathbb{Z}^\times, i = 1, \dots, n$). Recall that for each $\alpha \in \widetilde{\Delta}_+^{\mathrm{re}}$, x_α is defined by applying some braid group operators corresponding to an initial of the reduced expression $\prod_{k<0} s_{i_k}$, or $\prod_{k \geq 0} s_{i_k}$ (see §4). By the same proof as [DC–K–P, Proposition 2] we have:

Lemma 5.3.1. *Let x'_α, $x'_{-\alpha}$ ($\alpha \in \widetilde{\Delta}_+$) be defined as in (1.6.1) and §5.2 where $\prod_{k \geq 0} s_{i_k}$ (resp. $\prod_{k<0} s_{i_k}$)* (see (1.6.1)) *is replaced by a new expression obtained by substituting an arbitrary set of braid relations. Then $x'_\alpha, x'_{-\alpha}$ generate Z_ε over $Z_\varepsilon^{0,\mathrm{im}}$.* $\square$

Lemma 5.3.2. *The derivations $P_{z_{\alpha_i} x_i}$ and $-d_i z_{\alpha_i} f_i$ coincide on $x_{k\delta}^{(j)}$ ($j = 1, \dots, n$).*

Proof. We reduce this to the rank two calculation as follows. Consider the map M_i defined by modifying $\pi_-^-, \pi_-^+, \pi_+^-$ and π_+^+ in (5.3.3) to contain only factors corresponding to root vectors of the form $\pm\alpha_i \pm k\delta$. Then $M_i = \overline{M} B_i$ where $\overline{M}$ is as in (5.3.3) for the case of $\widetilde{\mathfrak{sl}_2}$ and

$$B_i = \exp\Big(\sum_{j \neq i, k > 0} x_{k\delta}^{(j)} t^k \omega_j^\vee\Big). \tag{5.3.4}$$

It follows from (5.3.1) that both $P_{z_{\alpha_i} x_i}$ and $-d_i z_{\alpha_i} f_i$ act as 0 on B_i, and therefore they coincide on M_i if they coincide on M.

In order to prove the lemma we must show that $P_{z_{\alpha_i}x_i}$ and $-d_i z_{\alpha_i} f_i$ coincide on Spec Z_ε when we consider $-d_i z_{\alpha_i} f_i$ when pulled back via the map M. Consider the subvariety $\Omega^{(i)}$ of Ω defined by replacing G in 4.1 by the connected subgroup of G whose Lie algebra is $\mathfrak{h} \oplus \mathbb{C}e_i \oplus \mathbb{C}f_i$. Then $\Omega^{(i)}$ is characterized as the subvariety of Ω for which $x_\alpha(p) = 0$ when $\alpha \neq \pm\alpha_i + k\delta$ $(p \in \Omega,\ k \in \mathbb{Z},\ \alpha \in \widetilde{\Delta}^{\rm re})$. Let Spec $Z_\varepsilon^{(i)}$ be the Poisson subvariety of Spec Z_ε consisting of $p \in$ Spec Z_ε for which $x_\alpha(p) = 0$ when $\alpha \neq \pm\alpha_i + k\delta$ $(p \in \Omega,\ k \in \mathbb{Z},\ \alpha \in \widetilde{\Delta}^{\rm re})$. Then $\gamma \circ \pi_{|\text{Spec } Z_\varepsilon^{(i)}} = M_i$. Using Frobenius it is clear that $Z_\varepsilon^{(i)}$ is a Poisson subalgebra of Z_ε. Since $\Omega^{(i)}$ and Spec $Z_\varepsilon^{(i)}$ are respectively invariant when integrating along f_i and $P_{z_{\alpha_i}x_i}$ the lemma follows. $\square$

Since the derivations $P_{z_{\alpha_i}x_i}$ and $-d_i z_{\alpha_i} f_i$ coincide on $Z_\varepsilon^{0,\rm im}$ by the rank 2 calculations, it suffices to show that

(5.3.5) $$P_{z_{\alpha_i}x_i}(x_\alpha) = -d_i z_{\alpha_i} f_i(x_\alpha),\ \alpha \in \widetilde{\Delta}^{\rm re},\ i = 1, \dots, n.$$

We give a proof assuming that $\alpha = \bar\alpha + k\delta$ where $k \geq 0$ and $\bar\alpha \in \Delta_+$. The cases where $\alpha \in \widetilde{\Delta}_-^{\rm re}$ or $\bar\alpha \in \Delta_-$ are similar. Given two non–proportional roots α and β, we denote by $R_{\alpha,\beta}$ the intersection of the $\mathbb{Z}$–span of α and β with $\widetilde{\Delta}^{\rm re}$ and let $R^+_{\alpha,\beta} = R_{\alpha,\beta} \cap \widetilde{\Delta}^{\rm re}_+$. Then $R_{\alpha,\beta}$ is a rank 2 root system with $R^+_{\alpha,\beta}$ being a subset of positive roots.

We consider the following two possibilities for α:

(a) $\alpha = \alpha_i + k\delta$, or

(b) α and α_i generate a subroot system of finite type.

In case (a) the statement follows from the $U_q(\widetilde{\mathfrak{sl}_2})$ calculations in the next section and the Drinfel′d relations (1.4.5) which show these calculations hold for all $i = 1, \dots, n$. Assume case (b) holds. There exist two simple roots, $\alpha_1, \alpha_2 \in \Pi$ and there exists $y \in Q^\vee,\ w \in W_0$ such that

$$ywR^+_{\alpha_1,\alpha_2} = R^+_{\alpha_i,\alpha} \text{ and } yw(\alpha_1) = \alpha_i.$$

Fix a reduced expression of $w = s_{i_1} \dots s_{i_k}$. Let $w_0' = s_1 s_2 s_1 \dots s_\varepsilon$, where $\varepsilon = 1$ or 2, be the reduced expression J' of the longest element of the Weyl group of R_{α_1,α_2} and let $m = \ell(w_0')$. Then the expression $ww_0' = s_{i_1} \dots s_{i_k} s_1 s_2 \dots s_\varepsilon$ is reduced. By [Pa, Proposition 7] it is possible to complete the reduced expression yww_0 to a reduced expression of some positive power of x^{-1} (where x is as in 1.4). Then $\widetilde{\Delta}^{\rm re}_+$ breaks into five pieces:

(5.3.6)
$$\begin{aligned}&R^1 := \{\beta_0, \dots, \beta_{-k}\},\ \beta_{-k-1} = \alpha_i,\\ &R^2 := \{\beta_{-k-2}, \dots, \beta_{-k-m}\} = R^+_{\alpha_i,\alpha},\ R^3 := \{\beta_{-k-m-1}, \dots, k\delta, \dots, \beta_l, l > 0\}.\end{aligned}$$

Let $\mathfrak{g}^i_\pm = \mathfrak{h} \otimes \mathbb{Z}[t, t^{-1}] \bigoplus_{\gamma \in R^i} \mathbb{C}e_{\pm\gamma},\ i = 1, 2, 3$. These are subalgebras of the Lie algebra $\mathfrak{g}$ normalized by the 3–dimensional subalgebra $\mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i$. This is so because it follows from [Pa, Theorem 1] that $R^i \pm \alpha_i \subset R^i$, for $i = 1, 2, 3$. Let $U^i_\pm$ be the subgroups of $U_\pm$ corresponding to the $\mathfrak{g}^i_\pm$.

We turn now to the map M which we decompose according to the above decomposition of $\widetilde{\Delta}^{\rm re}_+$:

$$\begin{aligned}M &= \pi_-^+ \times \pi_-^0 \pi_-^3 (\exp x_{-\alpha_i}))\pi_-^1 \times (\pi_0^0)^2 \pi_+^1 (\exp x_{\alpha_i} e_i) \pi_+^2 \pi_+^3 \pi_+^0 \pi_+^-\\ &= \pi_-^+ \times \pi_-^0 \pi_-^3 \pi_-^{1\prime} (\exp x_{-\alpha_i})) \times (\pi_0^0)^2 (\exp x_{\alpha_i} e_i) \pi_+^{1\prime} \pi_+^2 \pi_+^3 \pi_+^0 \pi_+^-,\end{aligned}$$

where $\pi_-^{1\prime} = (\exp x_{-\alpha_i} e_{-\alpha_i})\pi_-^1(\exp -x_{-\alpha_i} e_{-\alpha_i}) \subset U_-^1$ and $\pi_+^{1\prime} = (\exp x_i e_i)\pi_+^1(\exp -x_i e_i) \in U_+^1$.

Consider the subalgebra $Z_0^{1,2}$ of Z_ε generated over Z_0^0 by all x_γ and $x_{-\gamma}$ with $\gamma \in R^+_{\alpha_1,\alpha_2}$. We want to prove the following formula:

$$(d_i z_i f_i)(T_w(a)) = T_w(d_1 z_1 f_1(a)) \text{ for } a \in Z_0^{1,2}. \tag{5.3.7}$$

This formula implies (5.3.5) using the calculations in the rank 2 case, we have for $a \in Z_0^{1,2}$: $P_{z_{\alpha_i} x_i}(T_w(a)) = T_w P_{z_{\alpha_1} x_1}(a) = T_w(z_1 f_1(a))$.

In order to prove (5.3.7) note that the action of $z_i f_i$ on Z_ε may be calculated as follows. Write for $t \in \mathbb{C}$:

$$(\exp t z_i f_i)\pi(\exp -t z_i f_i) = \prod_k (\exp x_{-\beta_k}(t) e_{-\beta_k}) \prod_k (\exp x_{\beta_k}(t) e_{\beta_k}).$$

Then $f_i(x_{\beta_k}) = \frac{d}{dt} x_{\beta_k}(t)|_{t=0}$ $\beta \in \widetilde{\Delta}_+^{\mathrm{re}}$, and similarly for $x_{-\beta_k}$.

But x_α (resp. $x_{-\alpha}$) occurs only in π_2^+ (resp. π_2^-) and all other factors of π lie in the subgroups normalized by $\exp t z_i f_i$ and having trivial intersection with U_+^2 (resp. U_-^2). Thus, it suffices to perform the calculation in U_+^2 (resp. U_-^2). We have:

$$\begin{aligned}\prod_{s=k+2}^{k+m} \exp x_\beta^J(t) e_{\beta_s}^J &= (\exp t z_i f_i) \prod_{s=2}^{m} \exp T_w(x_s^{J'} e_s^{J'})(\exp -t z_i f_i) \\ &= T_w((\exp t z_1 f_1)(\prod_{s=2}^{m} \exp x_s^{J'} e_s^{J'})(\exp -t z_1 f_1)),\end{aligned}$$

and we can use the calculation in the finite type rank 2 case in [DC–K–P]. $\square$

5.4 In this section we make explicit the calculations for $U_q(\widetilde{\mathfrak{sl}_2})$, and $U_q^i(\widetilde{\mathfrak{sl}_2})$ which will imply Theorem 5.3.

In the case of $\widetilde{\mathfrak{sl}_2}$ the convex order (1.1.1) is as follows:

$$\delta - \alpha < 2\delta - \alpha < \cdots < 2\delta < \delta < \cdots < \alpha + \delta < \alpha. \tag{5.4.1}$$

Let $h_\pm(t) = \exp(\frac{1}{2}\sum_{k=1}^\infty x_{\pm k\delta} t^{\pm k})$, where $x_{k\delta} = x_{k\delta}^{(1)}$. Then the map (5.2.10) can be written as follows:

$$\Bigg(\begin{pmatrix} z_\omega^{-1} & 0 \\ 0 & z_\omega \end{pmatrix}\begin{pmatrix} 1 & \sum_{k=1}^\infty z_\alpha^2 x_{-k\delta+\alpha} t^{-k} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_-(t)^{-1} & 0 \\ 0 & h_-(t) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \sum_{k=0}^\infty x_{-\alpha-k\delta} t^{-k} & 1 \end{pmatrix}, \tag{5.4.2}$$

$$\begin{pmatrix} z_\omega & 0 \\ 0 & z_\omega^{-1} \end{pmatrix}\begin{pmatrix} 1 & \sum_{k=0}^\infty -x_{\alpha+k\delta} t^k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h_+(t) & 0 \\ 0 & h_+(t)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \sum_{k=1}^\infty -x_{-\alpha+k\delta} t^k & 1 \end{pmatrix}\Bigg).$$

Pulling this map back to $\overline{U}_1$ via the Frobenius map we can consider the corresponding map to (5.1.2):

$$\Bigg(\begin{pmatrix} K_\omega^{-1} & 0 \\ 0 & K_\omega \end{pmatrix}\begin{pmatrix} 1 & -K_\alpha X_{\le -1}^+ \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{\Phi} & 0 \\ 0 & \sqrt{\Phi}^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -X_{\le 0}^- & 1 \end{pmatrix},$$

$$(5.4.3)\qquad \left(\begin{pmatrix} K_\omega & 0 \\ 0 & K_\omega^{-1}\end{pmatrix}\begin{pmatrix} 1 & K_\alpha^{-1}X^+_{\geq 0} \\ 0 & 1\end{pmatrix}\begin{pmatrix} \sqrt{\Psi} & 0 \\ 0 & \sqrt{\Psi}^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ X^-_{>0} & 1\end{pmatrix}\right).$$

Note that (5.4.3) can be rewritten as

$$(5.4.4)\qquad \left(\begin{pmatrix} 1 & -X^+_{\leq -1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} K_\omega^{-1}\sqrt{\Phi} & 0 \\ 0 & K_\omega\sqrt{\Phi}^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -X^-_{\leq 0} & 1\end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & X^+_{\geq 0} \\ 0 & 1\end{pmatrix}\begin{pmatrix} K_\omega\sqrt{\Psi} & 0 \\ 0 & K_\omega^{-1}\sqrt{\Psi}^{-1}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ X^-_{>0} & 1\end{pmatrix}\right)$$

(5.4.5)

$$= \left(\begin{pmatrix} K_\omega^{-1}\sqrt{\Phi} + K_\omega\sqrt{\Phi}^{-1}X^-_{\leq 0}X^+_{\leq -1} & -K_\omega\sqrt{\Phi}^{-1}X^+_{\leq -1} \\ -K_\omega\sqrt{\Phi}^{-1}X^-_{\leq 0} & K_\omega\sqrt{\Phi}^{-1}\end{pmatrix},\right.$$

$$\left.\begin{pmatrix} K_\omega\sqrt{\Psi} + K_\omega^{-1}\sqrt{\Psi}^{-1}X^+_{\geq 0}X^-_{>0} & K_\omega^{-1}\sqrt{\Psi}^{-1}X^+_{\geq 0} \\ K_\omega^{-1}\sqrt{\Psi}^{-1}X^-_{>0} & K_\omega^{-1}\sqrt{\Psi}^{-1}\end{pmatrix}\right).$$

We consider the composition of map (5.3.3) with Fr^{-1} where $\gamma(a,b) = a^{-1}b$. In the case of $\widetilde{\mathfrak{sl}_2}$ we have:

$$A = \begin{pmatrix} K_1\sqrt{\Psi}\sqrt{\Phi}^{-1} + \sqrt{\Phi}^{-1}\sqrt{\Psi}^{-1}X^+X^-_{>0} & \sqrt{\Psi}^{-1}\sqrt{\Phi}^{-1}X^+ \\ \begin{gathered} K_1\sqrt{\Phi}^{-1}\sqrt{\Psi}X^-_{\leq 0} + K_1^{-1}\sqrt{\Phi}\sqrt{\Psi}^{-1}X^-_{>0} \\ +\sqrt{\Phi}^{-1}\sqrt{\Psi}^{-1}X^-_{\leq 0}X^-_{>0}X^+ \end{gathered} & K_1^{-1}\sqrt{\Phi}\sqrt{\Psi}^{-1} + \sqrt{\Phi}^{-1}\sqrt{\Psi}^{-1}X^+X^-_{\leq 0}\end{pmatrix}$$

Then using the Frobenius map to reinterpret (5.3.2) for $i = 0,\ 1$, we have $P_{-\overline{E}_i} = -d_iK_if_i,\ P_{\overline{F}_iK_i} = d_iK_ie_i$, where we let $K_0 = K_1^{-1}$.

Proposition 5.4.

$$\begin{aligned} &a)\ \ \{\overline{E}_1, A\} = -K_1[e_{21}, A], && \{\overline{E}_0, A\} = -K_0[t^{-1}e_{12}, A], \\ &b)\ \ \{\overline{F}_1K_1, A\} = K_1[e_{12}, A], && \{\overline{F}_0K_0, A\} = K_0[te_{21}, A], \\ &c)\ \ \{K_1, A\} = \frac{1}{2}K_1[e_{11} - e_{22}, A], && \{K_0, A\} = \frac{1}{2}K_0[e_{22} - e_{11}, A]. \end{aligned}$$

Proof. a), b) and c) follow from the formulas (3.3.1) by explicit calculation. □

§6 On the parametrization of finite–dimensional irreducible representations.

6.1 We recall the following material which is useful for studying finite dimensional representations of $\widetilde{U}_\varepsilon$.

Lemma 6.1 (cf. [S, ch IV]). *Consider the polynomial $Q(x) = 1 + a_1x + \dots a_dx_d$ over $\mathbb{C}$ of degree d. Then for a sequence of complex $N \times N$ matrices λ_n ($n \in \mathbb{Z}_+$) the following three conditions are equivalent:*

(1) *For all $s \in \mathbb{Z}_+$*

$$(6.1.1)\qquad \lambda_{s+d} + a_1\lambda_{s+d-1} + \dots + a_d\lambda_s = 0$$

(2) $\sum_{n=0}^{\infty} \lambda_n t^n = \frac{P(t)}{Q(t)}$, *where* $P(t)$ *is an* $N \times N$ *matrix polynomial with entries of degree* $< d$,

(3) *For all* $s \in \mathbb{Z}_+$

$$\lambda_s = \sum_{i=1}^{k} P_i(s)\gamma_i^s, \tag{6.1.2}$$

where $Q(x) = \prod_{i=1}^{k}(1-\gamma_i x)^{m_i}$, *the numbers* γ_i *are distinct and* $P_i(x)$ *is an* $N \times N$ *matrix polynomial with entries of degree* $< m_i$ $(i = 1 \ldots k)$.

Furthermore, for a sequence of complex $N \times N$ *matrices* λ_n $(n \in \mathbb{Z})$ *the conditions* (6.1.1) *and* (6.1.2) *for all* $s \in \mathbb{Z}$ *are equivalent and they imply*

$$\sum_{n=1}^{\infty} \lambda_{-n} t^n = -\frac{P(t^{-1})}{Q(t^{-1})} \tag{6.1.3}$$

Proof. The proof is the same as in [S] for the $N = 1$ case. □

Given a positive odd integer ℓ, define a *Frobenius map* $R \to R^F$ on the set of rational functions $\mathbb{C}(t)$ as follows. We may write $R = c\prod_i (t-a_i)^{m_i}$ where $c, a_i \in \mathbb{C}, m_i \in \mathbb{Z}$; then we let $R^F = c^\ell \prod_i (t - a_i^\ell)^{m_i}$. It is clear that this map is multiplicative (but not additive). It follows that we have

$$R^F(t^\ell) = \prod_{\eta \in \mu_\ell} R(\eta t), \tag{6.1.4}$$

where μ_ℓ denotes the set of all ℓ–th roots of 1.

6.2 We turn now to the study of finite–dimensional irreducible $\widetilde{U}_\varepsilon$–modules. Denote by $\mathfrak{a}_+$, $\mathfrak{a}_-$, and $\mathfrak{a}_0$ the subalgebras of $\widetilde{U}_\varepsilon$ generated by all elements $E_{\alpha+n\delta}$ $(\alpha \in \Delta_+, n \in \mathbb{Z})$; $E_{-\alpha+n\delta}$ $(\alpha \in \Delta_+, n \in \mathbb{Z})$; and K_α $(\alpha \in P)$, $E_{k\delta}^{(i)}$ $(k \in \mathbb{Z}^\times, i = 1, \ldots, n)$ respectively. Then by the PBW theorem we obtain:

$$\widetilde{U}_\varepsilon = \mathfrak{a}_- \otimes \mathfrak{a}_0 \otimes \mathfrak{a}_+ = \mathfrak{a}_- \otimes \mathfrak{a}_+ \otimes \mathfrak{a}_0. \tag{6.2.1}$$

Indeed the first equality in (6.2.1) follows from Proposition 1.7; the second equality follows from the first one using the fact that $\mathfrak{a}_+$ is generated by the $E_{\alpha_i+n\delta}$ and the relation (1.6.5*b*).

Proposition 6.2. *Let* V *be an irreducible* $\widetilde{U}_\varepsilon$*–module. Then* V *is finite–dimensional if and only if the following two conditions hold:*

(1) *There exists a common eigenvector for all the* K_α $(\alpha \in P)$ *and* $\psi_k^{(i)}$ $(i = 1, \ldots, n; k \in \mathbb{Z}^\times)$.

(2) *For each* $i = 1, \ldots, n$ *and each sign* $\pm$ *there exist* $c_1^\pm, \ldots, c_N^\pm \in \mathbb{C}$*, not all zero, such that one has in* V*:*

$$\sum_{j=1}^{N} c_j^\pm E_{\pm\alpha_i+(j+s)\delta} = 0 \text{ for all } s \in \mathbb{Z}. \tag{6.2.2}$$

Proof. Assume V is finite–dimensional. Then (1) is clear since all the K_α and $\psi_k^{(i)}$ mutually commute. Furthermore, (6.2.2) holds for $s=0$; taking brackets of this with $E_\delta^{(i)}$ or $E_{-\delta}^{(i)}$ $|s|$ times gives (6.2.2) for all s (due to (1.6.5b)).

Conversely, if (1) holds, then $V=\mathfrak{a}_-\mathfrak{a}_+v$ for some $v\in V$, due to the second equality of (6.2.1). Assume also that (2) holds. We write the elements of $\mathfrak{a}_\pm$ as (non–commutative) polynomials in the $E_{\pm\alpha_i+m\delta}$ $(i=1,\dots,n;m\in\mathbb{Z})$. Due to (6.2.2) and Proposition 1.7(c), after bringing an element of $\mathfrak{a}_-\mathfrak{a}_+v$ to a PBW form, only finitely many root vectors $E_{\pm\alpha+m\delta}$ $(\alpha\in\Delta_+,m\in\mathbb{Z})$ appear. Since the ℓ–th powers are scalars, we deduce that $\dim\mathfrak{a}_-\mathfrak{a}_+v<\infty$. $\square$

Remark 6.2. Taking bracket of both sides of (6.2.2) with $E_{\mp\alpha_i}$ and using (1.6.5d) we obtain:

(6.2.3)
$$\sum_{j=1}^N c_j^\pm\widehat{\psi}_{j+s}^{(i)}=0 \text{ for all } s\in\mathbb{Z},$$

where we let

(6.2.4)
$$\widehat{\psi}_m^{(i)}=K_i^{\mathrm{sgn}(m)}\psi_m^{(i)}.$$

Let V be a $\widetilde{U}_\varepsilon$–module and let $v_\lambda\in V$ be a common eigenvector of all the K_α $(\alpha\in P)$ and $\psi_k^{(i)}$ $(k\in\mathbb{Z}^\times,i=1,\dots,n)$. We have:

(6.2.5)
$$K_\alpha v_\lambda=\lambda_0(\alpha)v_\lambda,\ \text{where } \lambda_0:P\to\mathbb{C}^\times \text{ is a homomorphism.}$$

(6.2.6)
$$\psi_k^{(i)}v_\lambda=\lambda_k^{(i)}v_\lambda\ (k\in\mathbb{Z}^\times,i=1,\dots,n),\ \text{where } \lambda_k^{(i)}\in\mathbb{C}.$$

The collection λ consisting of a homomorphism $\lambda_0:P\to\mathbb{C}^\times$ and n sequences $\{\lambda_k^{(i)}\}_{k\in\mathbb{Z}^\times}$ $(i=1,\dots,n)$ is called the *weight* of v_λ, and v_λ is called a *weight vector* of the $\widetilde{U}_\varepsilon$–module V. It is convenient to introduce the generating series

$$\lambda_\pm^{(i)}(t)=\lambda_0(\alpha_i)^{\pm1}(1\pm(\varepsilon^{d_i}-\varepsilon^{-d_i})\sum_{s=1}^\infty\lambda_{\pm s}^{(i)}t^s).$$

Then the weight is given by the collection $(\lambda_0,\lambda_+^{(i)}(t),\lambda_-^{(i)}(t))$. The following lemma is immediate from Remark 6.2 and Lemma 6.1.

Lemma 6.2. *Let v_λ be a weight vector of a finite–dimensional $\widetilde{U}_\varepsilon$–module V. Then its weight $\lambda=(\lambda_0,\lambda_+^{(i)}(t),\lambda_-^{(i)}(t))$ satisfies the following two conditions $(i=1,\dots,n)$:*

(1) $\lambda_+^{(i)}(t)$ *is a rational function such that*

(6.2.8)
$$\lambda_+^{(i)}(0)=\lambda_0(\alpha_i),\quad \lambda_+^{(i)}(\infty)=\lambda_0(\alpha_i)^{-1}.$$

(2) $\lambda_-^{(i)}(t^{-1})=\lambda_+^{(i)}(t)$. $\square$

6.3 In this subsection we study the "diagonal" $\widetilde{U}_\varepsilon$–modules.

Definition 6.3 (a) An irreducible $\widetilde{U}_\varepsilon$–module V is called *diagonal* if all the scalars E_α^ℓ $(\alpha\in\widetilde{\Delta}^{\mathrm{re}})$ are zero.

(b) A weight vector v_λ of a $\widetilde{U}_\varepsilon$–module V is called *singular* if $E_{\alpha+k\delta}v_\lambda=0$ for all $\alpha\in\Delta_+,k\in\mathbb{Z}$.

Theorem 6.3 (a) *Any two singular vectors of a diagonal $\widetilde{U}_\varepsilon$–module are proportional.*

(b) *If two diagonal $\widetilde{U}_\varepsilon$–modules admit singular vectors, then these modules are isomorphic if and only if the weights of the singular vectors coincide.*

(c) *A diagonal $\widetilde{U}_\varepsilon$–module is finite dimensional if and only if it admits a singular vector v_λ and its weight $\lambda = (\lambda_0, \lambda_+^{(i)}(t), \lambda_-^{(i)}(t))$ satisfies the conditions (1) and (2) of Lemma 6.2.*

Proof. The proof of (a) and (b) and of the fact that a finite–dimensional diagonal $\widetilde{U}_\varepsilon$–module admits a singular vector follows from (6.2.2) in the usual way. The "only if" part of (c) follows from Lemma 6.2. Suppose now that v_λ is a singular vector of an irreducible $\widetilde{U}_\varepsilon$–module V and that λ satisfies conditions (1) and (2) of Lemma 6.2. Then by Lemma 6.1 we have for some $d_i \in \mathbb{Z}_+$ and $a_j^{(i)} \in \mathbb{C}$

$$(\widehat{\psi}^{(i)}_{s+d_i} + a_1^{(i)}\widehat{\psi}^{(i)}_{s+d_i-1} + \cdots + a^{(i)}_{d_i}\widehat{\psi}^{(i)}_s)v_\lambda = 0 \text{ for all } s \in \mathbb{Z}. \tag{6.3.1}$$

Since v_λ is singular, using (1.6.4d), (6.3.1) gives for each $i, j = 1, \ldots, n$ and $s \in \mathbb{Z}$:

$$E_{\alpha_j+(s-1)\delta}(E_{-\alpha_i+(d_i+1)\delta} + a_1^{(i)}E_{-\alpha_i+d_i\delta} + \cdots + a^{(i)}_{d_i}E_{-\alpha_i+\delta})v_\lambda = 0. \tag{6.3.2}$$

In other words, the vector

$$v_i' := (E_{-\alpha_i+(d_i+1)\delta} + a_1^{(i)}E_{-\alpha_i+d_i\delta} + \cdots + a^{(i)}_{d_i}E_{-\alpha_i+\delta})v_\lambda$$

is annihilated by all the operators $E_{\alpha_j+s\delta}$ where $j = 1, \ldots, n, s \in \mathbb{Z}$. Since these elements generate $\mathfrak{a}_+$, it follows from (a) and (6.3.2) that $v_i' = 0$ for all $i = 1, \ldots, n$. Applying to the previous equality $E^{(i)}_{\pm\delta}$ $|m|$ times we get by (1.6.5b) for all $m \in \mathbb{Z}$:

$$(E_{-\alpha_i+m\delta} + a_1^{(i)}E_{-\alpha_i+(m-1)\delta} + \cdots + a^{(i)}_{d_i}E_{-\alpha_i+(m-d_i+1)\delta})v_\lambda = 0. \tag{6.3.3}$$

Now due to the first equality in (6.2.1) we have:

$$V = \widetilde{U}_\varepsilon v_\lambda = \mathfrak{a}_- v_\lambda.$$

It follows from (6.3.3) that dim $V < \infty$ in the same way as at the end of the proof of Proposition 6.2. $\square$

Given a finite–dimensional $\widetilde{U}_\varepsilon$–module the weight of its singular vector is call its *highest weight.*

Corollary 6.3. *Finite–dimensional diagonal $\widetilde{U}_\varepsilon$–modules are in one to one correspondence with $(n+1)$–tuples $(\lambda_0, R_1(t), R_2(t), \ldots, R_n(t))$, where $\lambda_0 : P \to \mathbb{C}^\times$ is a homomorphism and $R_i(t)$ are rational functions such that*

$$R_i(0) = \lambda_0(\alpha_i), \quad R_i(\infty) = \lambda_0(\alpha_i)^{-1}. \tag{6.3.4}$$

The highest weight assocated to an $(n+1)$–tuple $(\lambda_0, R_1, \dots, R_n)$ is $\lambda = (\lambda_0, \lambda_+^{(i)}(t), \lambda_-^{(i)}(t))$, where $\lambda_+^{(i)}(t) = \lambda_-^{(i)}(t^{-1}) = R_i(t)$. $\square$

Remark 6.3. If $\lambda = (\lambda_0, \lambda_+^{(i)}(t), \lambda_-^{(i)}(t))$ and $\lambda' = (\lambda'_0, {\lambda'}_+^{(i)}(t), {\lambda'}_-^{(i)}(t))$ are the highest weights of diagonal finite–dimensional irreducible representations, then $\lambda\lambda'$ (where the multiplication is coordinate–wise) is the highest weight of $V \otimes W \cong W \otimes V$. This follows from the formula

$$\Delta(\hat{\psi}_k) = \sum_{j=0}^{k} \hat{\psi}_j \otimes \hat{\psi}_{k-j} \bmod (\mathfrak{a}_- \otimes \mathfrak{a}_+ + \mathfrak{a}_+ \otimes \mathfrak{a}_-),$$

which can be derived from [Be]. In the $\widetilde{\mathfrak{sl}_2}$ case, this is shown in [CP].

6.4 Denote by Spec $\widetilde{U}_\varepsilon$ the set of all finite–dimensional irreducible representations of the algebra $\widetilde{U}_\varepsilon$. By Schur's lemma we have the canonical map

$$\chi : \text{Spec } \widetilde{U}_\varepsilon \longrightarrow \text{Spec } Z_\varepsilon. \tag{6.4.1}$$

We now turn to the basic problem of calculating the image of χ, which we denote by $\mathcal{F}$. We shall identify the Poisson algebraic groups Spec Z_ε and Ω using the isomorphism π.

Given $\sigma \in$ Spec $\widetilde{U}_\varepsilon$ we call its image $\chi(\sigma) \in \Omega$ the *central character* of the representation σ. Note that the central character of any irreducible subquotient of $\sigma_1 \otimes \sigma_2$ ($\sigma_1, \sigma_2 \in$ Spec $\widetilde{U}_\varepsilon$) is equal to the product of central characters $\chi(\sigma_1)\chi(\sigma_2)$ in Ω. It follows that $\mathcal{F}$ is a subgroup of Ω.

Now note that we have canonical embeddings $i_0 : \mathbb{C}(t) \to \mathbb{C}((t))$ and $i_\infty : \mathbb{C}(t) \to \mathbb{C}((t^{-1}))$ obtained by expanding a rational function in a Laurent series at 0 and at ∞ respectively. Denote by $\mathbb{C}(t)_0$ the subalgebra of $\mathbb{C}(t)$ consisting of the rational functions that are regular at 0 and ∞. One has: $i_0(\mathbb{C}(t)_0) \subset \mathbb{C}[[t]]$, $i_\infty(\mathbb{C}(t)_0) \subset \mathbb{C}[[t^{-1}]]$. We shall identify $\mathbb{C}(t)$ and $\mathbb{C}(t)_0$ with their images under the maps i_0 and i_∞.

It is convenient to look at the groups $\underline{H}((t^{\pm 1}))$ by identifying them with the groups $\mathbb{C}((t^{\pm 1}))^\times \otimes_{\mathbb{Z}} Q^\vee$ (where $n \in \mathbb{Z}$ acts on $a \in \mathbb{C}((t^{\pm 1}))^\times$ as $a \mapsto a^n$ and on $\lambda \in Q^\vee$ as $\lambda \mapsto n\lambda$). Given a finite–dimensional representation of $\mathfrak{g}$ in a vector space V, an element $\sum_i a_i \otimes \lambda_i \in \underline{H}((t^{\pm 1}))$ acts on a weight vector $v_\mu \in V$ by the formula:

$$(\sum_i a_i \otimes \lambda_i) v_\mu = \prod_i a_i^{(\lambda_i | \mu)}.$$

Note that one has:

$$\underline{H}[[t^{\pm 1}]] = \mathbb{C}[[t^{\pm 1}]]^\times \otimes_{\mathbb{Z}} Q^\vee = \mathbb{C}[[t^{\pm 1}]]^\times \otimes_{\mathbb{Z}} P. \tag{6.4.2}$$

Consider the subgroup $\tilde{H}_+ = \mathbb{C}(t)_0^\times \otimes P$ of $\underline{H}[[t]] \subset \underline{G}[[t]]$ obtained using i_0 and the subgroup $\tilde{H}_- = \mathbb{C}(t)_0^\times \otimes P$ of $\underline{H}[[t^{-1}]] \subset \underline{G}[[t^{-1}]]$ obtained using i_∞.

Denote by $\tilde{N}_+^{\text{rat}}$ the subgroup of $\underline{G}[[t]]$ generated by $h(t) \in \tilde{H}_+$ such that $h(0) = 1$, by $\exp a(t) e_\beta$ such that $a(t) \in i_0(\mathbb{C}(t)_0)$ and $\beta \in \Delta_+$, and by $\exp a(t) e_{-\beta}$ such that $a(t) \in i_0(\mathbb{C}(t)_0)$, $a(0) = 0$ and $\beta \in \Delta_+$. Similarly, denote by $\tilde{N}_-^{\text{rat}}$ the subgroup of $\underline{G}[[t^{-1}]]$ generated by $h(t^{-1}) \in \tilde{H}_-$ such that $h(\infty) = 1$, by $\exp a(t^{-1}) e_\beta$ such that $a(t^{-1}) \in i_\infty(\mathbb{C}(t)_0)$, $a(\infty) = 0$ and $\beta \in \Delta_+$, and by $\exp a(t^{-1}) e_{-\beta}$ such that $a(t^{-1}) \in i_\infty(\mathbb{C}(t)_0)$ and $\beta \in \Delta_+$.

Let

$$\Omega^{\mathrm{rat}} = \{(hu_-, h^{-1}u_+) \mid u_\pm \in \tilde{N}_\pm^{\mathrm{rat}},\ h \in H\}. \tag{6.4.3}$$

Remark 6.4. Denote by $\underline{G}((t^{\pm 1}))^{\mathrm{rat}}$ the subgroup of $\underline{G}((t^{\pm 1}))$ that consists of elements g such that Ad g on $\mathfrak{g}((t^{\pm 1}))$ is in the Chevalley basis a matrix with elements in $\mathbb{C}(t)_0$. Let $\tilde{\tilde{G}}_{\mathrm{rat}} = \underline{G}((t^{-1}))^{\mathrm{rat}} \times \underline{G}((t))^{\mathrm{rat}}$. Then $\Omega^{\mathrm{rat}} = \tilde{\tilde{G}}_{\mathrm{rat}} \cap \Omega$. This follows from the results of [A–S] on Chevalley groups over semilocal rings, since the algebra $\mathbb{C}(t)_0$ is semilocal.

Introduce the following subgroup of the group Ω^{rat} :

$$\Omega_0^{\mathrm{rat}} = \{(a,a) \mid \mathrm{Ad}a \in (Ad\underline{G})(\mathbb{C}(t)_0), a(0) = hn_+, a(\infty) = h^{-1}n_-,\ \in H, n_\pm \in N_\pm.\}$$

6.5 First, we calculate the image of the "diagonal" part of the map χ. This will give, in particular, the image of the set of diagonal finite–dimensional irreducible representations.

In (1.6.4) replace u by ηt where $\eta \in \mu_\ell$ to obtain:

$$\exp(\ (q^{d_i} - q^{-d_i}) \sum_{k=1}^{\infty} E_{k\delta}^{(i)} \eta^k t^k) = 1 + (q^{d_i} - q^{-d_i}) \sum_{k=1}^{\infty} \psi_k^{(i)} \eta^k t^k. \tag{6.5.1}$$

Multiplying all equalities (6.5.1) over $\eta \in \mu_\ell$ we obtain after specializing to $\widetilde{U}_\varepsilon$:

$$\exp(\sum_{k=1}^{\infty} x_{k\delta}^{(i)} t^{k\ell}) = \prod_{\eta \in \mu_\ell} (1 + (\varepsilon^{d_i} - \varepsilon^{-d_i}) \sum_{k=1}^{\infty} \psi_k^{(i)} (\eta t)^k). \tag{6.5.2}$$

In a similar fashion we obtain

$$\exp(-\sum_{k=1}^{\infty} x_{-k\delta}^{(i)} t^{k\ell}) = \prod_{\eta \in \mu_\ell} (1 - (\varepsilon^{d_i} - \varepsilon^{-d_i}) \sum_{k=1}^{\infty} \psi_{-k}^{(i)} (\eta t)^k). \tag{6.5.3}$$

Consider a finite–dimensional irreducible representation σ of $\widetilde{U}_\varepsilon$ in a vector space V and let $v_\lambda \in V$ be a vector of weight λ. Applying both sides of (6.5.2) and (6.5.3) to v_λ, we obtain

$$\exp(\pm \sum_{k=1}^{\infty} x_{\pm k\delta}^{(i)} (\chi(\sigma)) t^k) = z_{\alpha_i}^{\mp 1} (\chi(\sigma)) \lambda_\pm^{(i)}(t)^F. \tag{6.5.4}$$

Hence we have

$$\pi_\pm^0 = \sum_{i=1}^{n} z_i^{\mp 1} (\lambda_+^{(i)}(t)^F) \otimes \omega_i^\vee. \tag{6.5.5}$$

by Lemma 6.2 (2). In other words we have (see (5.2.2)):

$$\pi_0^{0\,\pm 1} \pi_\pm^0 = \sum_{i=1}^{n} \lambda_+^{(i)}(t)^F \otimes \omega_i^\vee \in \underline{H}(Z_-^0 \otimes Z_0^0 \otimes Z_+^0). \tag{6.5.6}$$

Remark 6.5. For the study of finite dimensional representations it is convenient to write the map $\pi : \operatorname{Spec} Z_\varepsilon \to \Omega$ in the form $\pi = (\pi'_-(u'), \pi'_+(u'))$ where

$$\pi'_- = \prod_{k>0} \exp(z_{\beta_k} x_{-\beta_k} e_{-\beta_k}) \prod_{k<0} \exp(-z_{\alpha_i}^{-1} x_{-k\delta} t^{-k} \omega_i^\vee) \prod_{k\le 0} \exp(x_{-\beta_k} e_{-\beta_k}),$$
$$\pi'_+ = \prod_{k\le 0} \exp(-z_{\beta_k} x_{\beta_k} e_{\beta_k}) \prod_{k>0} \exp(z_{\alpha_i} x_{k\delta} t^k \omega_i^\vee) \prod_{k>0} \exp(-x_{\beta_k} e_{\beta_k}).$$

In this way the factors $\pi_0^0(h)^{\pm 1}$ in (5.3.7) are incorporated into $\pi'_\pm$ (see Remark 5.2). This allows us to make the identification between the rational functions in the first and second components of Ω.

Let $\tilde{H}_0$ be the subgroup of $\tilde{\Omega}_0^{\mathrm{rat}}$ consisting of elements of the form $(i_\infty(h(t)), i_0(h(t)))$, where $h(t) = \sum_i R_i(t) \otimes \omega_i^\vee,\ R_i(t) \in \mathbb{C}(t)_0$ and $R_i(0)R_i(\infty) = 1$.

Proposition 6.5. *The image under the map χ of the set of all diagonal irreducible finite–dimensional representations is $\tilde{H}_0$. The image of the representation with highest weight $(\lambda_0, \lambda_+^{(i)}(t), \lambda_-^{(i)}(t))$ is $(h(t), h(t))$ with $h(t) = \sum_i \lambda_+^{(i)}(t)^F \otimes \omega_i^\vee$.*

Proof. The proposition follows from (6.5.6) and Corollary 6.3. □

We also have the following nice corollary of (6.5.4):

Corollary 6.5. *The rational functions $\lambda_\pm^{(i)}(t)^F$ are independent of the choice of the weight λ of a finite–dimensional irreducible representation of $\widetilde{U}_\varepsilon$.* □

6.6 Recall that the definition of the real root vectors is based on the reduced expression $x = s_{j_1} \dots s_{j_d}$. Consider the set of root vectors

$$S = \{T_{i_d}^{-1} \dots T_{i_{k+1}}^{-1} E_{i_k} \mid 0 \le k \le d\}. \tag{6.6.1}$$

It is clear that every real root vector as defined in (1.6.1) is some power of T_x applied to a unique element of S. For each $\alpha \in \widetilde{\Delta}_+^{\mathrm{re}}$ fix $w = s_{i_1} s_{i_2} \dots s_{i_{k-1}} y$ where $1 \le i_j \le n$ $y \in Q^\vee$ and $\alpha_{i_k} \in \widetilde{\Pi}$ so that $w(\alpha_{i_k}) = \alpha$. Define $E'_{\pm\alpha} = T_w(E_{\pm\alpha_i})$. By [Be, Proposition 6.1 and Proposition 2.3] we have for each $E_s \in S$

$$E_s = P(E'_\alpha, h_{k\delta}^{(i)}),\ 1 \le i \le n,\ \alpha \in \widetilde{\Delta}^{\mathrm{re}} \tag{6.6.2}$$

where P is a polynomial.

Lemma 6.6.1. *Let V be a finite dimensional representation of $\widetilde{U}_\varepsilon$. For $\alpha \in \Delta$ the elements $\{E'_{\alpha+j\delta} = T_{i_1} T_{i_2} \dots T_{i_{k-1}} T^j_{\omega^\vee_{i_k}} E_{i_k} \mid j \in \mathbb{Z}\}$ act on V in such a way that there exist $c_1, \dots, c_N \in \mathbb{C}$, not all zero, such that one has in V:*

$$\sum_{j=1}^N c_j^\pm E'_{\pm\alpha+(j+s)\delta} = 0 \textit{ for all } s \in \mathbb{Z}. \tag{6.6.3}$$

Proof. This is proved in a similar manner to Proposition 6.2 where the element $E^{(i)}_{\pm\delta}$ is replaced by $T_{i_1} T_{i_2} \dots T_{i_{k-1}} T^d_{\omega^\vee_{i_k}} E^{(i_k)}_{\pm\delta}$. □

Lemma 6.6.2. *Let V be a finite dimensional $\widetilde{U}_\varepsilon$–module. Then the roots vectors $\{E_{\alpha+k\langle\alpha,x\rangle\delta} \mid k \in \mathbb{Z}\}$ act in a quasipolynomial manner on V (i.e. their matrix entries are finite linear combinations of functions of the form $P(k)\lambda^k$ where P is a polynomial and $\lambda \in \mathbb{C}$).*

Proof. Through each root vector $E_\alpha \in S$ consider set $\{T_x^k E_\alpha = E_{\alpha-k\langle\alpha,x\rangle\delta} \mid k \in \mathbb{Z}\}$. These sets exhaust all possible roots. Each E_α for $\alpha \in S$ is some polyonomial in the E'_α and imaginary root vectors (6.6.2). Since W normalizes $P^\vee$ we can assume $T_xT_{i_1}\dots T_{i_{k-1}} = T_{i_1}\dots T_{i_{k-1}}T_y$ for some $y \in P^\vee$. If $y = \prod_{m=1}^n \omega_m^{\vee d_m}$ then $T_y = \prod_m T_{\omega_m^\vee}^{d_m}$ and $T_yT_{\omega_{i_k}^\vee}^{d} E_{i_k} = T_{\omega_{i_k}^\vee}^{d+d_{i_k}} E_{i_k}$. Therefore $T_xE'_\alpha = E'_{\alpha+k'_{x\alpha}\delta}$ for all $\alpha \in \widetilde{\Delta}^{\rm re}$ where $k'_{x\alpha}$ and $d_{i_k} \in \mathbb{Z}$ is dependent on x and α. Using (6.6.2) and the fact that $T_x(h_k^{(i)}) = h_k^{(i)}$ we see $T_x^k(E_\alpha) = E_{\alpha-k\langle\alpha,x\rangle\delta} = P(E'_{\alpha+k'_{x\alpha}\delta}, h_{k\delta}^{(i)})$ where $k \in \mathbb{Z}$ and P are independent of $\mathbb{Z}$. From here using (6.6.3) the lemma follows. $\square$

Our main result is

Theorem 6.6. $\mathcal{F} = \Omega_0^{\rm rat}$.

Proof. First we show that $\mathcal{F} \subset \Omega_0^{\rm rat}$. By the definition of the map π and in view of Remark 6.5 and (6.5.6), it suffices to prove the following

Lemma 6.6.3. *Entries of the matrices $\pi_+^+,\ \pi_+^-,\ \pi_-^+,\ \pi_-^-$ lie in $Z_\varepsilon \otimes_{\mathbb{C}} \mathbb{C}(t)_0$.*

Proof. We shall consider the π_+^+, the proof in the remaining three cases being similar. Recall that

$$\pi_+^+ = \prod_{k\le 0,<} \exp(x_{\beta_k} e_{\beta_k}).$$

Using the commutation formula [St, Lemma 15], we may reorder this product such that it turns into a product of expressions of the form

$$\begin{aligned}\exp(a &\sum_{0\le j_1\le\dots\le j_s} x_{\beta_1+j_1\delta}\dots x_{\beta_s+j_s\delta} e_{\beta_1+\dots+\beta_s+(j_1+j_2\dots+j_s)\delta}) \\ &= \exp(a \sum_{0\le j_1\le\dots\le j_s} (x_{\beta_1+j_1\delta}t^{j_1})\dots(x_{\beta_s+j_s\delta}t^{j_s}) e_{\beta_1+\dots+\beta_s}),\end{aligned} \tag{6.6.4}$$

where $\beta_1,\dots,\beta_s \in \Delta_+$ and $a \in \mathbb{C}$.

Note that by Lemma 6.6.2 and Lemma 6.1 the entries of the matrices of elements $E_{\beta+k\delta}$ in a finite–dimensional representation σ of $\widetilde{U}_\varepsilon$ are quasipolynomials in k (i.e. a finite linear combination of functions of the form $P(k)\lambda^k$, where P is a polynomial and $\lambda \in \mathbb{C}$). Hence the same is true for $E_{\beta+k\delta}$. If σ is irreducible then all eigenvalues of $E_{\beta+k\delta}^\ell$ are equal to $x_{\beta+k\delta}(\chi(\sigma))$. Since their sum is the trace of $E_{\beta+k\delta}^\ell$, we conclude that $x_{\beta+k\delta}(\chi(\sigma))$ is a quasipolynomial in k. It follows by Lemma 6.1 that the entries of the matrices (6.6.4) lie in $\mathbb{C}(t)_0$. $\square$

In order to prove the inclusion $\mathcal{F} \supset \Omega_0^{\rm rat}$, we make the following two observations:

$$\mathcal{F} \supset \tilde{H}_0 \tag{6.6.5}$$

by Proposition 6.5(a), and

$$(a,a)\in\mathcal{F}\Longrightarrow \text{ the connected component of } K\cdot(a,a)\cap\Omega \text{ containing } (a,a) \text{ lies in } \mathcal{F}. \tag{6.6.6}$$

Observation (6.6.6) follows from Proposition 4.1 and the remark in [DC–K–P3, §4.1] which says, in particular, that if $\sigma\in\operatorname{Spec}\widetilde{U}_\varepsilon$, so that $\pi(\sigma)\in\mathcal{F}$, then the whole symplectic leaf of $\pi(\sigma)$ lies in $\mathcal{F}$.

Since Ω_0^{rat} is generated by $\tilde{H}_0$ and the 1–parameter root subgroups as descibed in §6.4, in view of (6.6.5) it suffices to show that for a given $a\in t\mathbb{C}(t)_0$ there exists $\beta,\gamma\in\mathbb{C}[t,t^{-1}]$ and $h\in\mathbb{C}(t)_0^\times$ such that

$$\begin{pmatrix}h^{-1}&0\\0&h\end{pmatrix}\begin{pmatrix}1&0\\\beta&1\end{pmatrix}\begin{pmatrix}h&0\\0&h^{-1}\end{pmatrix}\begin{pmatrix}1&0\\\gamma&1\end{pmatrix}=\begin{pmatrix}1&0\\a&1\end{pmatrix}.$$

Equivalently, given $a\in t\mathbb{C}(t)_0$ we wish to find $\beta,\gamma\in\mathbb{C}[t,t^{-1}]$ and $h\in\mathbb{C}(t)_0$ such that

$$a=h^2\beta+\gamma. \tag{6.6.7}$$

This is straightforward. $\square$

6.7 For $\widetilde{U}_\varepsilon=U_\varepsilon(\widetilde{\mathfrak{sl}_2})$ Theorem 6.6 means the following. Let $g(t)=-\sum_{k=0}^\infty z_\alpha x_{\alpha+k\delta}t^k,\ h(t)=z_\alpha\exp(\sum_{k=1}^\infty x_{k\delta}t^k)=z_\alpha(1+(\varepsilon-\varepsilon^{-1})\sum_{k=1}^\infty\psi_k t^k)^F,\ f(t)=-\sum_{k=1}^\infty x_{-\alpha+k\delta}t^k$. Using Lemma 6.1 it follows that $\mathcal{F}=\{(A(t),A(t))\}$ where

$$A(t)=\begin{pmatrix}1&g(t)\\0&1\end{pmatrix}\begin{pmatrix}h(t)^{1/2}&0\\0&h(t)^{-1/2}\end{pmatrix}\begin{pmatrix}1&0\\f(t)&1\end{pmatrix}. \tag{6.7.1}$$

We shall now calculate the image under χ of an evaluation representation. We let $E=E_1,\ F=F_1,\ K=K_\alpha$. Recall [DC–K] that the center of $U_\varepsilon(\mathfrak{sl}_2)$ is generated by the elements $x=(\varepsilon-\varepsilon^{-1})^\ell E^\ell,\ y=(\varepsilon-\varepsilon^{-1})^\ell F^\ell,\ z_1=K_\omega^\ell,\ c=(\varepsilon-\varepsilon^{-1})^2FE+K\varepsilon+K^{-1}\varepsilon^{-1}$. Let $z=z_1^2$.

Recall that due to Jimbo [J2] there exists for $a\in\mathbb{C}^\times$ an "evaluation" homomorphism $ev_a:U_q(\widetilde{\mathfrak{sl}_2})\to U_q(\mathfrak{sl}_2)$ given by

$$ev_a(E_{-\alpha+k\delta})=(aq^{-1})^kK^kF,$$
$$ev_a(E_{\alpha+k\delta})=(aq^{-1})^kEK^k.$$

One can show that:

$$ev_a(1+(q-q^{-1})\sum_{k=1}^\infty\psi_i t^k)=\frac{a^2q^{-4}(t^2-a^{-1}q^2c_qt+a^{-2}q^4)}{(1-aq^{-3}Kt)(1-aq^{-1}Kt)}$$

where $c_q=(q-q^{-1})^2FE+qK+q^{-1}K^{-1}$.

Consider a finite–dimensional irreducible representation σ of $U_\varepsilon(\mathfrak{s}l_2)$ with the prescribed values of central elements $x,\ y,\ z$, and c. Then $\tilde{\sigma}_a := \sigma \circ ev_a \in \operatorname{Spec} U_\varepsilon(\widetilde{\mathfrak{s}l_2})$. We have $\chi(\tilde{\sigma}_a) = (g(t), g(t))$, where

$$g(t) = \begin{pmatrix} 1 & -\frac{xz}{1-a^\ell zt} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h(t)^{1/2} & 0 \\ 0 & h(t)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{zya^\ell t}{1-a^\ell zt} & 1 \end{pmatrix} \tag{6.7.2}$$

Here $h(t) = \frac{(1-a^\ell c_1^\ell t)(1-a^\ell c_1^{-\ell} t)}{(1-a^\ell zt)^2}$, where $c = c_1 + c_1^{-1}$. Note that if $x = 0$ then $h(t) = \frac{1-a^\ell z^{-1} t}{1-a^\ell zt}$.

It follows that taking central characters of tensor products of all $\tilde{\sigma}_a$ where σ is a diagonal representation of $U_\varepsilon(\mathfrak{s}l_2)$, we get the whole subgroup $\tilde{H}_0$ of Ω_0^{rat}. Furthermore, taking all $\tilde{\sigma}_a$ with σ having central character $x, y = 0, z = 1$ (resp. $x = 0,\ y,\ z = 1$), we get all matrices

$$\begin{pmatrix} 1 & \frac{\alpha}{1+\beta t} \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ \frac{\alpha t}{1+\beta t} & 1 \end{pmatrix}.$$

Since these matrices together with $\tilde{H}_0$ generate the group Ω_0^{rat} [AS], we obtain

Proposition 6.7. *The image under χ of the set of all irreducible subquotients of tensor products of evaluation representations of $U_\varepsilon(\widetilde{\mathfrak{s}l_2})$ coincides with Ω_0^{rat}.* $\square$

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